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Using the radon transform to solve inclusion problems in elasticity

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Abstract

From the Fourier transform method, the modified Green operator integral over a bounded domain in an infinite elastic medium takes, on each domain point, the form of a weighted average over an angular distribution of a single elementary operator. The Radon transform provides a geometric definition of the weight function characteristic of the domain shape, in terms of the domain intersections with all planes passing through the point. It allows a geometrically more meaningful analytical resolution of the general inclusion problem in an infinite medium of general elasticity symmetry, the "inclusion" being any bounded domain possibly made of groups (or distributions) of inclusions. The method is also likely to provide insights in the related problem of effective moduli estimates for heterogeneous microstructures. The determination of the weight functions characteristics of the involved inclusional domain shapes is therefore a key step of the resolution, the mean values of these weight functions being of first-order interest. Here, it is exemplified, on the case of cuboidal domain shapes, that for material morphologies involving shapes of hardly accessible exact mean weight functions, one can make use of approximate (conveniently analytical) expressions, to remain more accurate than using ellipsoidal approximations of the shapes.

Keywords: Radon transform; Green operator integral; Eshelby inclusion problem; Elasticity

1. Introduction

The problem of the elastic behaviour of an isolated inclusion in an infinite elastic matrix submitted to a uniform strain from infinity has been solved by Eshelby (1957) in the case of an ellipsoidal shape. For this particular shape type, the problem simplifies owing to the fact that the involved so called "modified Green operator integral" (and the related Eshelby tensor) over the inclusion is uniform in it. Since then, this property of ellipsoidal shapes is widely used in inclusion-related problems, aiming at stress–strain fields, stored energy, or overall property estimates for heterogeneous materials (see for example reviews in Mura, 1982 or Nemat-Nasser and Hori, 1999). Applications to real materials are made possible in approximating,

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more or less abusively, as ellipsoidal the shapes of contained particles, voids or grains (in polycrystals). Even for ellipsoidal inclusions, except for isotropic and transversally isotropic elasticity, fully analytical calculations of the modified Green operator integral are not at hand (see Withers (1989) for transverse isotropy for example), and the calculations in general elasticity symmetry are mostly performed using the Fourier transform method with numerical integration techniques. For non-isolated inclusions, calculations are furthermore complicated by interactions terms, which also have been approximated in considering pairs of ellipsoidal inclusions (as for example in Berveiller et al. (1987), from the Green/Fourier approach, in the context of polycrystals elasto-plasticity). Increasing difficulties are encountered when considering non-ellipsoidal inclusion shapes, for which the involved modified Green operator integral (or the Eshelby tensor) over them is not uniform in general. Such situations are so far treated in mixing partly analytical and partly numerical calculations, with case per case analyses depending on both the considered shape and the elasticity symmetry. As examples of non-ellipsoidal inclusion types that have been considered in different inclusion-related problems, in either contexts of Green/Fourier techniques, potential methods, or numerical approaches, one finds cubes (Chiu, 1977), cubic inclusion pairs (Canova et al., 1992), finite cylinders (Wu and Du, 1995a,b), polyhedra (Rodin, 1996), polygonal axi-symmetric inclusions, and duplex (i.e. coated) inclusions (Riccardi and Montheillet, 1998, 1999), star shaped inclusions (Mura, 1997) among others. More general calculation methods, formally applicable for inclusions of general shapes, have also been proposed, based on different expansions or decompositions of the **G** Green operator (Kinoshita and Mura, 1971), or of its **Γ** related modified Green operator integral (Kneer, 1965; Willis, 1981), but mostly as an alternative to the Eshelby approach of the ellipsoidal shape case.

A better capacity to account for non-ellipsoidal inclusion shapes being likely to improve the resolution of many inclusion-related problems, we here reconsider the analytical calculation of the modified Green operator integral for the general inclusion shape case, for general elasticity symmetry, from the framework of the Radon transform method (Gel'fand et al., 1966), and with the help of integral geometry considerations (Santalo, 1976). The Radon transform method, of which recent papers present applications in various fields of mechanics and physics (Jiang, 2000; Lykotrakis and Georgiadis, 2003) advantageously provides in inclusion problems context a geometrically more meaningful analytical resolution method, which holds for general bounded domains possibly made of groups of inclusions. Based on this Radon transform method, the resolution of the "inclusion problem" mainly aims at calculating a shape-characteristic angular weight function at each **r** point of the considered domain, which is defined from the domain intersections with all planes passing through the **r** point. A particular interest of this method for the non-ellipsoidal, general, inclusion problem, that we here stress, is to allow an approximate resolution (when the exact one is not at easy hand) in terms of the approximation of the characteristic weight function, and more precisely of its mean value over the concerned domain. With regard to real inclusion shapes in materials, approximating these mean weight functions may be more relevant than best ellipsoidal fitting of inclusion shapes, in preserving tractable analytical resolutions. This may also be of use for other characteristic shapes of a material microstructure, namely characteristic shapes of spatial arrangements of inclusions.

In Section 2, we recall, *for a general bounded domain shape*, the expression of the modified Green operator integral, as classically resulting from the Fourier transform method. One then specifies the geometrical interpretation of the involved weight function with regard to the domain geometry, and its connection to the inverse Radon transform of the characteristic function of the domain. The applicability to bounded domains made of groups of inclusions, and from then to spatial arrangements of statistically defined symmetry, is illustrated from the case of an inclusion pair set. One next recall the expression of the involved elementary operator, which is known connected to the modified Green operator integral of the infinitely oblate spheroid (platelet), also characteristic of laminate structures (Walpole, 1967; Kinoshita and Mura, 1971). In Section 3, we reset the general problem of the isolated inclusion (bounded domain) in an infinite, elastically anisotropic, matrix, to illustrate in simple cases how the Radon transform method can be of use. It is also next illustrated for the related problem of the effective moduli estimate of a two-phase inclusion/

matrix microstructure with congruent inclusions uniformly distributed in space, according to some symmetry shape characteristic. In these simple examples, the mean weight functions related to the involved characteristic shapes are the needed morphological informations for resolution. In Section 4, calculations of such mean weight functions are exemplified for single convex cuboidal shapes in the whole range between the cube and its reciprocal octahedron, which all have the sphere as "best ellipsoidal fit". Exact ones are provided for the limit cube and octahedron cases, approximate ones for the intermediate cuboids. Related modified Green operator integrals are calculated for isotropic elasticity, to be compared to the sphere one. Section 5 concludes.

2. The modified Green operator integral for an inclusion

We consider an infinite homogeneous elastic medium of $\mathbf{C}$ elasticity moduli, of $\mathbf{G}(\mathbf{r}-\mathbf{r}')$ Green "strain" tensor, giving the $\mathbf{u}(\mathbf{r})$ displacement field due to a punctual force $\mathbf{F}(\mathbf{r}')$ by $u_i(\mathbf{r}) = G_{ij}(\mathbf{r}-\mathbf{r}')F_j(\mathbf{r}')$, classically defined (Hill, 1952; Kröner, 1958) by

$$C_{mnpq}G_{pj,qn}(\mathbf{r}-\mathbf{r}') + \boldsymbol{\delta}(\mathbf{r}-\mathbf{r}')\Delta_{mj} = 0. \tag{1}$$

In Eq. (1), Δ is the Kronecker tensor, $\boldsymbol{\delta}(\mathbf{r})$ is the delta (generalised) function in R^3 of arbitrary chosen frame, with $\mathbf{r} = (x_1, x_2, x_3)$. Introducing the notations

$$\Gamma_{pqjn}(\mathbf{r}-\mathbf{r}') = -\left(\left. \frac{\partial^2 G_{pj}(\mathbf{r}-\mathbf{r}')}{\partial x_q \partial x_n} \right|_{(p,q),(j,n)} \right)$$

with '$(p,q),(j,n)$' specifying the symmetry on the pairs of indices within brackets, the diagonal symmetric [1] $\Gamma_{pqjn}(\mathbf{r}-\mathbf{r}')$ operator, is generally called the modified Green operator. The integral of this operator over a bounded V domain in the considered medium, here denoted $t^V_{pqjn}(\mathbf{r})$, which depends on the V domain shape and on the $\mathbf{C}$ elasticity moduli of the matrix containing V, writes by definition

$$t^V_{pqjn}(\mathbf{r}) = \int_V \Gamma_{pqjn}(\mathbf{r}-\mathbf{r}')\,\mathrm{d}\mathbf{r}' \tag{2}$$

with V of general morphology so far. In this paper, $\mathbf{r}$ is always inside the concerned domain.

2.1. Calculation from the Fourier transform method

The calculation of $\mathbf{t}^V(\mathbf{r})$ from Eq. (1) can be performed from the Fourier transform [2] of Eq. (1)

$$C_{mnpq}\breve{G}_{pj}(\mathbf{K})k_q k_n = \Delta_{mj} \tag{3}$$

where $\mathbf{K} = (k_1, k_2, k_3)$ and $|\mathbf{K}| = k$. In spherical coordinates such that $k_i = k\omega_i$, $\mathbf{K} = k\boldsymbol{\omega}$, with $\omega_1 = \sin(\theta)\cos(\varphi)$, $\omega_2 = \sin(\theta)\sin(\varphi)$, $\omega_3 = \cos(\theta)$, Eq. (3) writes (ω standing for (θ, ϕ))

[1] That is (pq, jn) symmetric.

[2] For the Fourier and inverse Fourier transforms, we use the following definitions:

$$\breve{G}_{pj}(\mathbf{K}) = \int G_{pj}(r)\exp^{\mathrm{i}\mathbf{Kr}}\,\mathrm{d}r, \quad G_{pj}(\mathbf{r}) = \frac{1}{8\pi^3}\int \breve{G}_{pj}(\mathbf{K})\exp^{-\mathrm{i}\mathbf{Kr}}\,\mathrm{d}\mathbf{K}$$

$$\breve{\delta}(\mathbf{K}) = \int \delta(\mathbf{r})\exp^{\mathrm{i}\mathbf{Kr}}\,\mathrm{d}\mathbf{r} = 1, \quad \delta(\mathbf{r}) = \frac{1}{8\pi^3}\int \exp^{-\mathrm{i}\mathbf{Kr}}\,\mathrm{d}\mathbf{K}.$$

$$C_{mnpq}\omega_q\omega_n k^2\breve{G}_{pj}(\mathbf{K}) = M_{mp}(\boldsymbol{\omega})k^2\breve{G}_{pj}(\mathbf{K}) = \Delta_{mj} \Rightarrow k^2\breve{G}_{pj}(\mathbf{K}) = (M^{-1})_{pj}(\boldsymbol{\omega}). \tag{4}$$

The $k^2\breve{G}_{pj}(\mathbf{K})$ product in Eq. (4) is independent on the k modulus of the $\mathbf{K}$ vector. Next, in Eq. (2), one replaces $\Gamma_{pqjn}(\mathbf{r}-\mathbf{r}')$ by the inverse transform of its Fourier transform to write

$$t^V_{pqjn}(\mathbf{r}) = \frac{1}{8\pi^3}\int_V\left(\int ((M^{-1})_{pj}(\boldsymbol{\omega})\omega_q\omega_n)\Big|_{(pq),(jn)} \exp^{-i\mathbf{K}(\mathbf{r}-\mathbf{r}')}\,d\mathbf{K}\right)d\mathbf{r}' \tag{5a}$$

Writing $d\mathbf{K} = k^2\,dk\sin(\theta)\,d\theta\,d\varphi = k^2\,dk\,d\omega$, one obtains

$$t^V_{pqjn}(\mathbf{r}) = \frac{1}{8\pi^3}\int_V\left(\int_\Omega t^e_{pqjn}(\omega)\int_{k=0}^{\infty} k^2\exp^{-i\mathbf{K}(\mathbf{r}-\mathbf{r}')}\,dk\,d\omega\right)d\mathbf{r}' \tag{5b}$$

where Ω is the unit sphere. Permutation of the V and Ω integrals ends to formally write

$$t^V_{pqjn}(\mathbf{r}) = \int_\Omega t^e_{pqjn}(\omega)\psi_V(\omega,\mathbf{r})\,d\omega \tag{6}$$

with

$$\psi_V(\omega,\mathbf{r}) = \frac{1}{8\pi^3}\xi_V(\omega,\mathbf{r}),\quad \xi_V(\omega,\mathbf{r}) = \int_V\left(\int_{k=0}^{\infty} k^2\exp^{-ik\boldsymbol{\omega}(\mathbf{r}-\mathbf{r}')}\,dk\right)d\mathbf{r}' \tag{7a}$$

$$t^e_{pqjn}(\omega) = \left((M^{-1})_{pj}(\omega)\omega_q\omega_n\right)_{(pq),(jn)} \tag{7b}$$

Obviously from Eq. (6), and whatever the V domain shape is, $\mathbf{t}^V(\mathbf{r})$ writes under the form of a weighted angular average of $\mathbf{t}^e(\omega)$ elementary operators, with a $\psi_V(\omega,\mathbf{r})$ weight function. This form for $\mathbf{t}^V(\mathbf{r})$ can be found in many works related to inclusion or inclusion pairs problems (for examples, Berveiller et al., 1987; Canova et al., 1992), since it is on the way of the Green/Fourier resolution method. It is here attempted, in the context of inclusion-related problems, and especially for other situations than ellipsoidal domain shapes, to more fully take benefit of the fact that Eq. (6), with Eqs. (7), is the expression of $\mathbf{t}^V(\mathbf{r})$ in terms of its inverse Radon transform. We first and mainly make geometrically explicit the $\psi_V(\omega,\mathbf{r})$ weight function in this framework for general bounded (simply connected or not) V domain shapes. The expression of the (known) $\mathbf{t}^e(\omega)$ operator is next recalled.

2.2. Explicitation of the weight function

In a (x,y,z) frame with $Oz//\mathbf{K}//\boldsymbol{\omega}$, such that $\mathbf{K}\cdot\mathbf{r} = k\boldsymbol{\omega}\cdot\mathbf{r} = kz$, Eq. (7a) reads

$$\xi_V(\omega,\mathbf{r}) = -\frac{1}{2}\int_V\left(\int_{-\infty}^{+\infty}(it)^2\exp^{-it(z-z')}\,dt\right)d\mathbf{r}' = -\pi\int_V \delta''(z-z',\omega)\,d\mathbf{r}' \tag{8}$$

with $\delta''(z-z',\omega)$, the second z-derivative of the one-dimensional delta function, $\boldsymbol{\omega}$ both defining a direction in space and the infinite z-oriented axis along this direction. Setting $d\mathbf{r}' = ds_V(z',\omega)\,dz'$, with $s_V(z',\omega)$ the area of the section of the volume V by the plane of $z' = \boldsymbol{\omega}\cdot\mathbf{r}'$ equation, yields

$$\xi_V(\omega,\mathbf{r}) = -\pi\int_{z'=-\infty}^{z'=+\infty}\left(\int_{s_V(z',\omega)} ds_V(z',\omega)\right)\delta''(z-z',\omega)\,dz' = -\pi s''_V(z,\omega) \tag{9a}$$

where $s''_V(z,\omega)$ is the second z-derivative of $s_V(z,\omega)$. Eq. (7a) thus provides $\psi_V(\omega,\mathbf{r})$ as

$$\psi_V(\omega,\mathbf{r}) = -\frac{1}{8\pi^2}s''_V(z,\omega) \tag{9b}$$

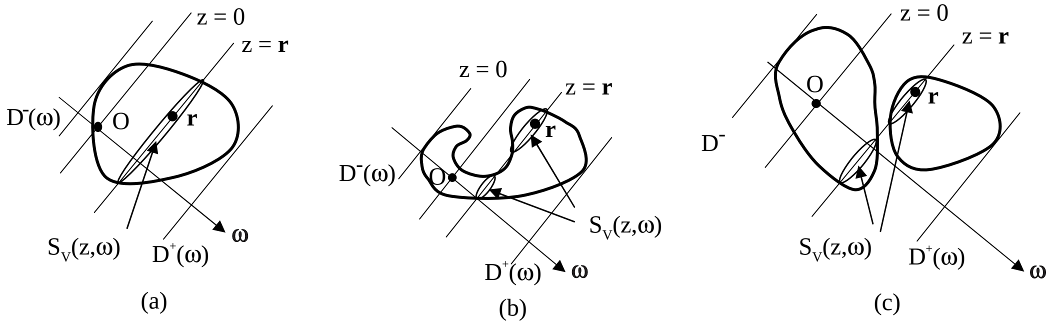

Fig. 1. **ω**-breadth and sections of **ω** normal for: (a) a simply connected and convex; (b) a simply connected and concave; (c) a multiply connected, regular domain.

i.e. proportional to the second z-derivative of the planar section area of V, of **ω** normal, passing through the **r** point. So far, the only restrictions on the shape of the V bounded domain are the existence conditions for the $s''_V(z,\omega)$ derivatives, for which it is enough to assume regular (smooth) V domains, in the sense of non-strictly zero product of the principal radii of curvature at each point of the V surface. Since V is bounded, the z' integral in Eq. (9a) is only non-zero within the $[D_V^-(\omega), D_V^+(\omega)] = 2D_V(\omega)$ breadth of V in the **ω**-direction, i.e. the distance between the two opposite tangent planes to V, of **ω**-normal. As assessed in what follows, these definitions likely hold as well for single domains or for sets of regular sub-domains, as illustrated in Fig. 1 for simply connected convex (a) or not convex V domains (b), and for a multiply connected V domain made of two sub-domains (c). For V simply connected convex domains, the 2 $D_V(\omega)$ breadths characterise the support function of V, otherwise they characterise the support function of the convex hull of V, while the z' integral over $\lfloor D_V^-(\omega), D_V^+(\omega)\rfloor$ then dissociates into several ones over separated intervals. For centrosymmetric domains, the z' integral symmetrically ranges over $\lfloor -D_V(\omega), D_V(\omega)\rfloor$. Special attention will be paid to simply connected, convex centrosymmetric, regular domains.

For general V domains, $\psi_V(\omega,\mathbf{r})$ is not uniform in V. Let's then consider the $\overline{\psi_V}(\omega)$ spatial mean value of $\psi_V(\omega,\mathbf{r})$ (i.e. $\overline{\xi_V}(\omega)$ of $\xi_V(\omega,\mathbf{r})$) over V, which appears in the $\overline{\mathbf{t}^V} = \frac{1}{v}\int_V \mathbf{t}^V(\mathbf{r})\,\mathrm{d}\mathbf{r}$ volume average of $\mathbf{t}^V(\mathbf{r})$ over V from Eq. (6). Setting $\mathrm{d}\mathbf{r} = \mathrm{d}s_V(z,\omega)\,\mathrm{d}z$, as for $\mathrm{d}\mathbf{r}'$ in $\xi_V(\omega,\mathbf{r})$, and with the area integral part directly written $s_V(z,\omega)$, one obtains [3] for $\overline{\xi_V}(\omega)$

$$\overline{\xi_V}(\omega) = -\frac{\pi}{v}\int_{D_V^-(\omega)}^{D_V^+(\omega)} s''_V(z,\omega)s_V(z,\omega)\,\mathrm{d}z = \frac{\pi}{v}\int_{D_V^-(\omega)}^{D_V^+(\omega)} (s'_V(z,\omega))^2\,\mathrm{d}z \tag{10}$$

with the right-hand side term of Eq. (10) resulting from part integration, since $s_V(D_V^-(\omega)) = s_V(D_V^+(\omega)) = 0$. It is known from the Radon transform theory (Gel'fand et al., 1966) that, in R^3, the **ω** integral of $\frac{1}{8\pi^2}\int_{D_V^-(\omega)}^{D_V^+(\omega)}(s'_V(z,\omega))^2\,\mathrm{d}z$ over the unit sphere provides the v volume of the V domain, in terms of its sections by planes, as

$$\frac{1}{8\pi^2}\int_\Omega\int_{D_V^-(\omega)}^{D_V^+(\omega)} (s'_V(z,\omega))^2\,\mathrm{d}z\,\mathrm{d}\omega = -\frac{1}{8\pi^2}\int_\Omega\left(\int_{D_V^-(\omega)}^{D_V^+(\omega)} s''_V(z,\omega)s_V(z,\omega)\,\mathrm{d}z\right)\mathrm{d}\omega = v \tag{11}$$

[3] In the sense of generalised functions.

This results, as recalled in Section 3, from the relation

$$\int X_V(\mathbf{r})X_V(\mathbf{r})\,\mathrm{d}\mathbf{r} = \int_V X_V(\mathbf{r})\,\mathrm{d}\mathbf{r} = v = -\frac{1}{8\pi^2}\int_\Omega\left(\int_{D_V^-(\omega)}^{D_V^+(\omega)} s''_V(z,\omega)s_V(z,\omega)\,\mathrm{d}z\right)\mathrm{d}\omega$$
$$= \frac{v}{8\pi^3}\int_\Omega \overline{\xi_V}(\omega)\,\mathrm{d}\omega \tag{12a}$$

where $X_V(\mathbf{r})$ is the characteristic function of the V domain ($X_V(\mathbf{r}) = 1$ for $\mathbf{r}$ in V and 0 if not). Thus $\int_\Omega \overline{\xi_V}(\omega)\,\mathrm{d}\omega = 8\pi^3$, and, from Eq. (7a), $\int_\Omega \overline{\psi_V}(\omega)\,\mathrm{d}\omega = 1$, as expected. Accordingly, one has

$$X_V(\mathbf{r}) = -\frac{1}{8\pi^2}\int_\Omega s''_V(z,\omega)\,\mathrm{d}\omega = \int_\Omega \psi_V(\omega,\mathbf{r})\,\mathrm{d}\omega = 1 \tag{12b}$$

known (also to be addressed later on) as the inverse Radon transform of the $X_V(\mathbf{r})$ characteristic function of V. Conversely, the $s_V(z,\omega)$ function is the Radon transform of $X_V(\mathbf{r})$. Eq. (12a) also writes

$$\int X_V(\mathbf{r})X_V(\mathbf{r})\,\mathrm{d}\mathbf{r} = \lim_{\mathbf{r}0\to 0}\left(\int X_V(\mathbf{r}-\mathbf{r}0)X_V(\mathbf{r}+\mathbf{r}0)\,\mathrm{d}r\right) = v \tag{13}$$

which is the global covariance function of a V deterministic compact set, [4] at the $\mathbf{r}0$ origin of the covariance space, say $C^V(\mathbf{r}0 = 0)$.

2.2.1. Single inclusions

Application of what preceeds to inclusions of general shape needs to consider each geometric case specifically. Ellipsoidal inclusions are the most particular cases of simply connected, convex centrosymmetric, regular domains, which all derive from the sphere case by linear transform. For a spherical V inclusion of radius R, planar section areas normal to any $\boldsymbol{\omega}$-direction identically write $s_V(z,\omega) = \pi(R^2 - z^2)$, such that $s''_V(z,\omega) = -2\pi$, $\xi_V(\omega,\mathbf{r}) = 2\pi^2 = \xi_V(\omega)$, and $\psi_V(\omega,\mathbf{r}) = \frac{1}{4\pi} = \psi_V(\omega)$, independently on $\mathbf{r}$ in V and on the $\boldsymbol{\omega}$-direction. It is easily verified that for an ellipsoidal V inclusion of volume v

$$s_V(z,\omega) = \frac{3v}{4D_V(\omega)}\left(1 - \left(\frac{z}{D_V(\omega)}\right)^2\right)$$

and that $s''_V(z,\omega) = -\frac{3v}{2D_V(\omega)^3}$, still independently on the $\mathbf{r}$ position in V and of the z value along a given $\boldsymbol{\omega}$-direction. Thus, for ellipsoids

$$\psi_V(\omega) = \left(\frac{3}{4\pi}\right)^2\left(\frac{v}{3D_V(\omega)^3}\right)$$

Note that for any convex V domain, the integral $\int_\Omega \frac{1}{3D_V(\omega)^3}\,\mathrm{d}\omega$ over the unit sphere equals $\int_\Omega \frac{R^3_{V^*}(\omega)}{3}\,\mathrm{d}\omega = v^*$, i.e. provides the v^* volume of the V^* reciprocal convex body of V (Santalo, 1976), since by definition, the distance (or radius vectors) function $R_{V^*}(\omega)$ of V^* equals $1/D_V(\omega)$, the inverse of the V support function. [5] Thus, for ellipsoids

$$\psi_V(\omega) = \frac{R_{V^*}(\omega)^3}{3v^*} = \frac{1}{3v^* D_V(\omega)^3}$$

Fig. 2 shows, for an ellipsoid, the shapes of the body, its support function, its weight function and its reciprocal body. Normal directions to planar parts of the V surface, if any, correspond to vertices of the V^*

[4] The measure of V eroded by the union of the extremities of the $\mathbf{r}0$ vector.

[5] The vv^* product fulfils $4^3/3! \leqslant vv^* \leqslant (4\pi/3)^2$, the right- (respectively, left-) hand side equality holding for reciprocal ellipsoids ($v = 4\pi abc/3$; $v^* = 4\pi/3abc$) (respectively, for reciprocal polyhedra).

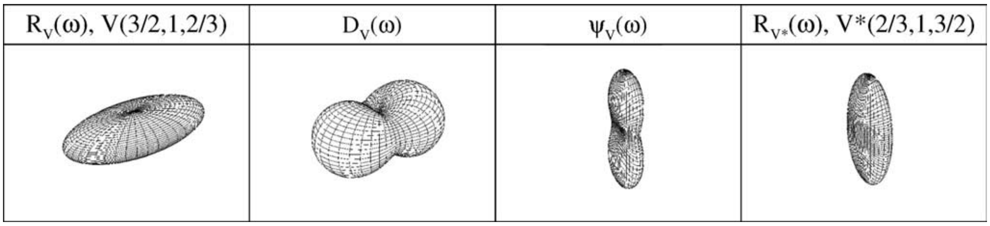

Fig. 2. From left to right, a V ellipsoid, its support function, weight function and V^* reciprocal body, at same scale.

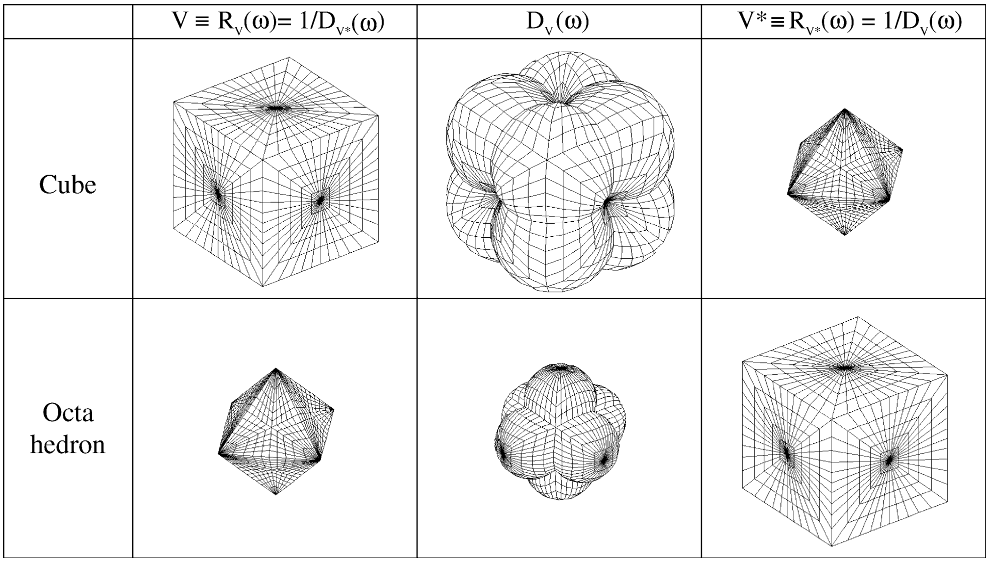

Fig. 3. From left to right, the V volume, its support function and its reciprocal V^* domain (at same scale), for the cube ($\max(|x_1|, |x_2|, |x_3|) = 1$), and for the octahedron ($|x_1| + |x_2| + |x_3| = 1$).

reciprocal body. Convex polyhedra, as the cube and the octahedron shown in Fig. 3, are limit cases of regular convex domains. Cuboidal inclusions of equation $\sum_{i=1}^{3}(x_i)^{2n} = 1$, for $n \in [0.5, \infty[$, of which the drawn octahedron ($n = 0.5$) and cube ($n = \infty$) are limit cases, will be considered in part 4, in comparison to the (unit) sphere case ($n = 1$).

2.2.2. Inclusion pairs

The Radon transform framework is valid for any type of regular bounded, possibly multiply connected, V domains (i.e. groups of inclusions). Note that although not considered here, it also applies to unbounded domains, the "area" of an infinite section needing to be thought of as a function of the intersecting plane (Gel'fand et al., 1966).

One can in particular directly apply it to inclusion pairs, V_i, V_j, such that $V = V_i \cup V_j$ and $V_i \cap V_j = \Phi$, the empty set Fig. 1c. Then, Eqs. (9a) and (10) become

$$\xi_V(\omega, \mathbf{r}) = -\pi(s''_{V_i}(z, \omega) + s''_{V_j}(z, \omega)) \tag{14a}$$

$$\overline{\xi_V}(\omega) = -\frac{\pi}{v}\int_{D_V^-(\omega)}^{D_V^+(\omega)} (s''_{V_i}(z,\omega) + s''_{V_j}(z,\omega))(s_{V_i}(z,\omega) + s_{V_j}(z,\omega))\,\mathrm{d}z$$
$$= \frac{\pi}{v}\int_{D_V^-(\omega)}^{D_V^+(\omega)} (s'_{V_i}(z,\omega) + s'_{V_j}(z,\omega))^2\,\mathrm{d}z \tag{14b}$$

with $v = v_i + v_j, D_V(\omega)$ being the support function of the convex hull of the inclusion pair, and the individual sections of V_i (respectively, V_j) and their derivatives being zero when the V intersection with the z plane is empty. Globally, $\overline{\xi_V}(\omega) = ((v_i/v)\overline{\xi_{V_i}}(\omega) + (v_j/v)\overline{\xi_{V_j}}(\omega)) + \overline{\xi_{V_i,V_j}}(\omega)$, which yields $\overline{\mathbf{t}^V} = (v_i/v)\overline{\mathbf{t}^{V_i}} + (v_j/(v)\overline{\mathbf{t}^{V_j}} + \overline{\mathbf{t}^{V_i,V_j}}$, with the same $\mathbf{t}^e(\omega)$ elementary operator involved as for a single inclusion. The $\overline{\xi_{V_i,V_j}}(\omega)$ cross pair interaction term, and its related $\overline{t^{V_i,V_j}}$ operator, specified later on, corresponds globally to the two involved (i/j) and (j/i) mean cross pair interaction operators in the inclusion pair problem, as treated for example in Berveiller et al. (1987) for ellipsoidal inclusions. The cross pair interaction term in $\overline{\xi_V}(\omega)$ is only non-zero in limited $\boldsymbol{\omega}$-angular sectors (along $\boldsymbol{\omega}$-directions for which $z = \mathbf{r}\cdot\boldsymbol{\omega}$ planes intersect both inclusions) which decrease with increasing distances between the inclusions. It has the property that $\int_\Omega \overline{\xi_{V_i,V_j}}(\omega)\,\mathrm{d}\omega = 0$, since both ω-integrals of $\overline{\xi_V}(\omega)$ and of $\sum_{\beta=i,j}(v\beta/v)\overline{\xi_{V\beta}}(\omega)$ equal $8\pi^3$ (due to $\int X_V(\mathbf{r})X_V(\mathbf{r})\,\mathrm{d}\mathbf{r} = \sum_{\beta=i,j}(\int X_{V\beta(\mathbf{r})X_{V\beta}(\mathbf{r})\,\mathrm{d}\mathbf{r}})$). Consider for example two V_1, V_2 inclusions, of O_1 and O_2 centres at, respectively, $-R$ and R distances of a O origin along some $\boldsymbol{\omega}_0$-axis, as exemplified in Fig. 4 for two ellipsoids. Let us write accordingly $\overline{\mathbf{t}^V}$ and $\overline{\xi_V}(\omega)$ as $\overline{\mathbf{t}^{V\omega0}}$ and $\overline{\xi_{V\omega0}}(\omega)$, to indicate the $\boldsymbol{\omega}_0$ dependency of such a $V = V_1 \cup V_2$ domain with regard to the relative inclusion positions. One can always write $\overline{\xi_{V\omega0}}(\omega) = \overline{\xi_{V_0}}(\omega) + \overline{\delta\xi_{V\omega0}}(\omega)$ (denoting $\overline{\xi_{V_0}}(\omega)$ for $\sum_{\alpha=1,2}(v\alpha/v)\overline{\xi_{V\alpha}}(\omega)$), where $\int_\Omega \overline{\delta\xi_{V\omega0}}(\omega)\,\mathrm{d}\omega = 0$. The $\overline{\delta\xi_{V\omega0}}(\omega)$ term writes from (the last form of) Eq. (14b)

$$\overline{\delta\xi_{V\omega0}}(\omega) = \frac{2\pi}{v_1+v_2}\int_{\max(\rho-D_{V_2},-D_{V_1}-\rho)}^{\min(\rho+D_{V_2},D_{V_1}-\rho)} (s'_{V_1}(z+\rho,\omega)s'_{V_2}(z-\rho,\omega))\,\mathrm{d}z \tag{15a}$$

with $\rho = R\cos(\omega_0,\omega)$. For ellipsoids, $s'_V(z,\omega)$ being z-linear, this gives, for $\rho > 0$ and for $\alpha \geqslant 1$ in taking $D_{V_1}(\omega) = \alpha D_{V_0}(\omega)$, $D_{V_2}(\omega) = \alpha^{-1}D_{V_0}(\omega)$, [6] and with $\eta = z/D_{V_0}$ and $u = \rho/D_{V_0}$

$$\overline{\delta\xi_{V\omega0}}(\omega) = \frac{2\pi v_1 v_2}{v_1+v_2}\left(\frac{-3}{2D_{V_0}(\omega)^2}\right)^2 D_{V_0}(\omega)\int_{\eta=u-\alpha^{-1}}^{\eta=\min(u+\alpha^{-1},\alpha-u)} (\eta^2 - u^2)\,\mathrm{d}\eta$$
$$= \frac{8\pi^3}{v}C^{V_1,V_2}(\omega)\begin{cases} \left(\frac{2}{3\alpha^3}\right) & \text{for } 0 < u < (\alpha-\alpha^{-1})/2 \\ \left(\frac{4u^3}{3} - u(\alpha^2+\alpha^{-2}) + \frac{1}{3}(\alpha^3+\alpha^{-3})\right) & \text{for } (\alpha-\alpha^{-1})/2 < u < (\alpha+\alpha^{-1})/2 \end{cases} \tag{15b}$$

$$C^{V_1,V_2}(\omega) = 4\pi^2 D_{V_0}(\omega)(R_1^{V_1}(\omega)R_2^{V_1}(\omega))^{1/2}(R_1^{V_2}(\omega)R_2^{V_2}(\omega))^{1/2}$$

This is the same result as the one obtained for the ellipsoidal inclusion pair in Berveiller et al. (1987). If the ellipsoidal geometry remains by far the simplest case for exact analytical resolution, with increasing but case per case tractable complexity, non-ellipsoidal inclusion pairs or larger patterns can be treated similarly, and a $\overline{\psi_V}(\omega)$ mean weight function be calculated from $\overline{\xi_V}(\omega)$.

[6] Or similar expressions when $\rho < 0$ or for $\alpha \leqslant 1$ in appropriately changing the integration intervals.

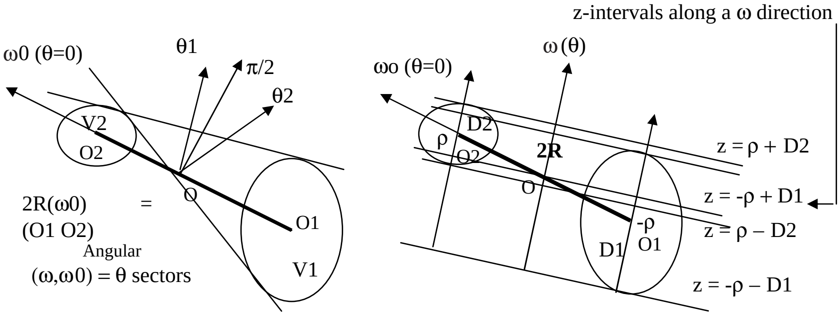

Fig. 4. A pair of ellipsoidal inclusions $V = V_1 \cup V_2$, and angular ($\boldsymbol{\omega}$, $\boldsymbol{\omega}_0$) sectors and $z(\boldsymbol{\omega})$ intervals for the calculation of $\overline{\xi_V}(\omega)$.

2.2.3. Averages of inclusion pair positions

Statistically considering a set of relative positions of two V_1, V_2 inclusion centres that maps some closed, centrosymmetric, S surface, a $\overline{\overline{(\xi_{V\omega0})}}^S(\omega)$ related average of $\overline{\xi_{V\omega0}}(\omega)$ over this S "pair position distribution" would write

$$\overline{\overline{(\xi_{V\omega0})}}^S(\omega) = \frac{1}{S}\int_S (\overline{\xi_{V_0}}(\omega) + \overline{\delta\xi_{V\omega0}}(\omega))\,\mathrm{d}S(\omega_0) = \overline{\xi_{V_0}}(\omega) + \overline{\overline{(\delta\xi_{V\omega0})}}^S(\omega) \tag{16a}$$

with, from Eq. (16a), and ρ now varying with $\boldsymbol{\omega}_0$, a mean cross pair interaction

$$\overline{\overline{(\delta\xi_{V\omega0})}}^S(\omega) \equiv \int_\Omega \left(\int_{z^-(\omega_0)}^{z^+(\omega_0)} (s'_{V_1}(z+\rho,\omega))(s'_{V_2}(z-\rho,\omega))\,\mathrm{d}z \right) \mathrm{d}S(\omega_0) \tag{16b}$$

The related $\overline{\overline{(\psi_{V\omega0})}}^S(\omega)$ average weight function therefore shares into a $\overline{\psi_{V_0}}(\omega)$ average self-interaction term, which is the one of a V_0 inclusion for congruent inclusion pairs, or of the form $\sum_\alpha (v\alpha/v\overline{\psi_{V\alpha}}(\omega)$ otherwise, and of a $\overline{\overline{(\delta\psi_{V\omega0})}}^S(\omega)$ average cross interaction term, both accessible to a geometrical calculation. We do not here enter the question of existing S surfaces yielding a null mean cross pair interaction term, for inclusion pairs of general V_1, V_2 shapes.

Note however, that from Eq. (16b) this is directly seen to be fulfilled for a pair of spherical inclusions with respect to an isotropic (spherical) surface of constant R radius ($\mathrm{d}(\boldsymbol{\omega}_0) \equiv \mathrm{d}\boldsymbol{\omega}_0 \forall \boldsymbol{\omega}_0$), noticing that the $\boldsymbol{\omega}_0$ integral then identifies to the $\boldsymbol{\omega}$ one, which is null. It is consequently also fulfilled, by linear transform, for all ellipsoidal congruent $V_1 \equiv V_2 \equiv V_0$ inclusion pairs over the inclusion-shape-homothetic ellipsoidal $S \equiv V_0$ surface. Furthermore, the property also directly appears to be fulfilled for any pair of (not necessarily congruent) ellipsoids over such an isotropic S surface, since from Eq. (15b), one has, with $\mathrm{d}\omega_0 = \mathrm{d}\cos(\theta_0)\,\mathrm{d}\phi_0 = \frac{\mathrm{d}\rho\mathrm{d}\phi_0}{R} = \frac{D_{V_0}(\omega)\,\mathrm{d}u\,\mathrm{d}\phi_0}{R}$, and the ϕ_0 integral only contributing 2π

$$\overline{\overline{(\delta\xi_{V\omega0})}}^S(\omega) \equiv \left(\int_0^{(\alpha-\alpha^{-1})/2} \left(\frac{2}{3\alpha^3}\right)\mathrm{d}u + \int_{(\alpha-\alpha^{-1})/2}^{(\alpha+\alpha^{-1})/2} \left(\frac{4u^3}{3} - u\left(\alpha^2+\alpha^{-2}\right) + \frac{1}{3}\left(\alpha^3+\alpha^{-3}\right)\right)\mathrm{d}u \right) = 0 \tag{17}$$

This consequently holds for any pair of ellipsoidal inclusions over any ellipsoidal S surface.

The cases of more general shapes could be addressed in following a similar way. For example, from Eq. (17), a pair of ellipsoids appear to behave as a sphere pair with regard to the $\boldsymbol{\omega}_0$-integration at fixed ω value. This suggests the property of null mean pair cross interaction term to be in general fulfilled for S surfaces, if any, that would make the V_1, V_2 inclusion pair seen as a sphere pair upon $\boldsymbol{\omega}_0$-integration.

2.3. Explicitation of the elementary operator

Each $\mathbf{t}^e(\omega) = \mathbf{t}^e(\theta, \phi)$ elementary operator in $\mathbf{t}^V$ ($\mathbf{r}$) from Eq. (6) is an axi-symmetric operator, defined in reference to the parallel planes of $\boldsymbol{\omega}$ normal direction in some reference medium frame. This operator is of since long identified characteristics, among which is the identity with the operator of infinitely flat oblate spheroids (platelets) of same $\boldsymbol{\omega}$ orientation of its small axis (Walpole, 1967; Kinoshita and Mura, 1971). The $\mathbf{M}$ tensor in Eq. (4) is the tensor which governs the propagation velocity of plane harmonic waves (Christensen, 1979). For general elasticity anisotropy, considering the (0,0) oriented $\mathbf{t}^e$ (0,0) elementary operator for $(\theta, \phi) = (0, 0)$, the only non-zero terms from Eq. (7b) correspond to $\boldsymbol{\omega} = (0, 0, 1)$, and the M_{mp} coefficients identify to the C_{m3p3} elastic moduli of the infinite medium, for $\mathbf{C}$ expressed in the operator axes frame, that we can momentarily denote $\mathbf{C}_{(0,0)}$. In this frame, $\mathbf{M}$ is therefore a 3×3 symmetric sub-matrix of $\mathbf{C}_{(0,0)}$, which is given Table 1. Thus, the t^{e}_{p3j3} (0,0) non-zero terms of the $\mathbf{t}^e$ (0,0) operator expressed in the operator frame make a symmetric 3×3 matrix as well, that we denote $\Delta\mathbf{t}$, and they are linked to the N_{pj} terms of the inverse $\mathbf{N} = \mathbf{M}^{-1}$ tensor as also given Table 1.

The $\mathbf{N}$ tensor is known as the tensor of Kelvin–Christoffel stiffnesses (Walpole, 1981). The elementary operators take this form (of a non-zero $\Delta\mathbf{t}$ block) when expressed in their specific frame, and in terms of the $\mathbf{C}$ elasticity moduli expressed in this frame as well. [7] When $\mathbf{C}$ is defined in some (0,0) reference frame as $\mathbf{C}_{(0,0)}$, the appropriate $\mathbf{C}_{(\theta,\phi)}$ moduli in the $t^e(\theta, \phi)$ operator frame are obtained, by using the $R(\theta, \varphi)$ rotation matrix given in Table 2. Now, since one needs to express all the $\mathbf{t}^e(\theta, \phi)$ elementary operators in a same matrix frame to calculate $\mathbf{t}^V$ ($\mathbf{r}$) in this frame, the one within which $\mathbf{C}$ is denoted $\mathbf{C}_{(0,0)}$ say, the expression, in this frame, of any (θ, ϕ)-oriented elementary operator is obtained in using the inverse $R^{-1}(\theta, \varphi) = R^{\mathrm{t}}(\theta, \varphi)$ rotation matrix.

For elastic isotropy, since in all frames $C_{3333} = \lambda + 2\mu = 2\mu(1-v)/(1-2v) = (K + 4\mu)/3$ and $C_{1313} = C_{2323} = \mu$, the $\mathbf{C}$ frame identification is made useless. The non-zero terms of $\Delta\mathbf{t}$ are $N_{11} = 1/C_{1313}$, $N_{22} = 1/C_{2323}$, $N_{33} = 1/C_{3333}$, and those of $\mathbf{t}^e(\theta, \phi)$ in the (θ, ϕ) operator frame are $t_{3333} = \frac{1-2v}{2\mu(1-v)} = \frac{3}{K+4\mu} = \frac{1}{\mu} - \frac{1}{2\mu(1-v)} = a$, $t_{((2,3),(2,3))} = t_{((3,1),(3,1))} = \frac{1}{4\mu} = b$, for all (θ, ϕ) orientations. In reference to a same (0,0), frame, the $\mathbf{t}^e(\theta, \phi) = \mathbf{t}^e(\omega)$ operators thus rearrange as

$$t^{\mathrm{e}}_{pqjn}(\omega) = a\delta^a_{pqjn}(\omega) + b\delta^b_{pqjn}(\omega) \tag{18a}$$

with the $\boldsymbol{\delta}^a$ and $\boldsymbol{\delta}^b$ (elasticity independent) operators writing

$$\delta^a_{pqjn}(\omega) = (\omega_j\omega_p\omega_n\omega_q)|_{(p,q),(j,n)}, \quad \delta^b_{pqjn}(\omega) = 4\big((\Delta_{jp} - \omega_j\omega_p)\omega_n\omega_q\big)|_{(p,q),(j,n)} \tag{18b}$$

The identity of form of the elementary $\mathbf{t}^e(\omega)$ operator and of the $\mathbf{t}^{\mathrm{P}(\omega)}$ platelet operator of same small axis orientation, can be simply established in considering two-phases laminate structures, as recalled in Appendix A, the material layers being considered as infinitely flat spheroids. This can also be directly assessed, in denoting that the weight function $\psi_{\mathrm{P}}(\omega) \equiv \frac{1}{D_{\mathrm{p}}(\omega)^3}$ of a $\boldsymbol{\omega}$-oriented platelet, seen as a flattened spheroid, is zero in any direction but $\boldsymbol{\omega}$ where it is infinite (since $D_{\mathrm{p}}(\omega) \to 0$ while $D_{\mathrm{p}}(\omega' \neq \omega) \to \infty$), what defines, together with a unit value of the weight function integral over the unit sphere, a delta-like function on the unit sphere, say

$$t^{P(\omega)}_{pqjn} \cong \int_{\Omega} t^{\mathrm{e}}_{pqjn}(\omega')\delta(\omega' - \omega)\,\mathrm{d}\omega' = t^{\mathrm{e}}_{pqjn}(\omega), \quad \text{with } \delta(\omega' - \omega) = \delta(\cos\theta' - \cos\theta)\delta(\phi' - \phi) \tag{19}$$

[7] The axisymmetry of the elementary operators is verified from $\mathbf{M}$ in Table 1, the constitutive $\mathbf{C}$ moduli of which do not change for a rotation around the 3 axis (i.e. the $\boldsymbol{\omega}$-direction) but only rotates accordingly. On the contrary, when the $\boldsymbol{\omega}$-direction is changed, the $\mathbf{C}$ matrix block constitutive of $\mathbf{M}$ changes as well.

Table 1
Left: expression of the **M** matrix, with the $\mathbf{C}_{(0,0)}$ medium moduli; right: the general $\Delta\mathbf{t}$ non-zero part of the $\mathbf{t}^{e}$ (0,0) operator, both expressed in the operator frame

$$\mathbf{M} = \begin{vmatrix} C_{1313} & C_{1323} & C_{1333} \\ C_{1323} & C_{2323} & C_{2333} \\ C_{1333} & C_{2333} & C_{3333} \end{vmatrix}_{0,0} ; \quad \Delta\mathbf{t} = \begin{vmatrix} t_{1313} & t_{1323} & t_{1333} \\ t_{1323} & t_{2323} & t_{2333} \\ t_{1333} & t_{2333} & t_{3333} \end{vmatrix} = \begin{vmatrix} \frac{N_{11}}{4} & \frac{N_{12}}{4} & \frac{N_{13}}{2} \\ \frac{N_{12}}{4} & \frac{N_{22}}{4} & \frac{N_{23}}{2} \\ \frac{N_{13}}{2} & \frac{N_{23}}{2} & N_{33} \end{vmatrix}$$

Table 2
$R(\theta,\varphi)$ rotation matrix from some (0,0) reference frame in (θ,φ) and $(\omega_1,\omega_2,\omega_3)$ notations

$$R(\theta,\varphi) = \begin{vmatrix} -\cos(\theta)\cos(\varphi) & \sin(\varphi) & \sin(\theta)\cos(\varphi) \\ -\cos(\theta)\sin(\varphi) & -\cos(\varphi) & \sin(\theta)\sin(\varphi) \\ \sin(\theta) & 0 & \cos(\theta) \end{vmatrix} = \begin{vmatrix} -\omega_1\omega_3/(1-\omega_3^2)^{1/2} & \omega_2/(1-\omega_3^2)^{1/2} & \omega_1 \\ -\omega_2\omega_3/(1-\omega_3^2)^{1/2} & -\omega_1/(1-\omega_3^2)^{1/2} & \omega_2 \\ (1-\omega_3^2)^{1/2} & 0 & \omega_3 \end{vmatrix}$$

The $\mathbf{t}^{e}(\omega)$ operators are now on denoted $\mathbf{t}^{P(\omega)}$, although $\mathbf{t}^{e}(\omega)$ is in fact the limit of the platelet operator form when its thickness to diameter ratio tends to zero. Note that among remaining differences that may matter according to specific situations, the platelet operator remains invertible while $\mathbf{t}^{e}(\omega)$ is not.

3. Inclusion problems from the Radon transform method

Using $\psi_V(\omega,\mathbf{r})$ from Eq. (9b), the $\mathbf{t}^{V}(\mathbf{r})$ modified Green operator integral at **r** points of a V domain given in Eq. (6) and its $\overline{\mathbf{t}^{V}}$ mean value over V have the inverse Radon transforms

$$t^{V}_{pqjn}(\mathbf{r}) = \int_{\Omega} t^{P(\omega)}_{pqjn}\psi_V(\omega,\mathbf{r})\,\mathrm{d}\omega = -\frac{1}{8\pi^2}\int_{\Omega} t^{P(\omega)}_{pqjn} s''_V(z,\omega)\,\mathrm{d}\omega \tag{20a}$$

$$\overline{t^{V}_{pqjn}} = \int_{\Omega} t^{P(\omega)}_{pqjn}\overline{\psi_V}(\omega)\,\mathrm{d}\omega = -\frac{1}{8\pi^2}\int_{\Omega} t^{P(\omega)}_{pqjn}\left(\frac{1}{v}\int_{D^-_V(\omega)}^{D^+_V(\omega)} s''_V(z,\omega)s_V(z,\omega)\,\mathrm{d}z\right)\mathrm{d}\omega \tag{20b}$$

This is next used in inclusion problems resolution, after few recalls about the Radon transform.

3.1. The Radon transform elementary properties

For any scalar or tensorial function $\mathbf{q}(\mathbf{r})$ entirely summable over R^3, the Radon transform RT[$\mathbf{q}(\mathbf{r})$] is defined (Gel'fand et al., 1966) as the integrals of $\mathbf{q}(\mathbf{r})$ over the $z = \boldsymbol{\omega}\cdot\mathbf{r}$ planes, that we will here denote RT[$\mathbf{q}(\mathbf{r})$] $= \mathbf{Q}(z,\omega)$. When $\mathbf{q}(\mathbf{r})$ is defined over some bounded V region, say $\mathbf{q}^{V}(\mathbf{r})$, the Radon transform RT[$\mathbf{q}^{V}(\mathbf{r})$] is defined as the integrals of $\mathbf{q}^{V}(\mathbf{r})$ on the s_V (z,ω) sections of V by the $z = \boldsymbol{\omega}\cdot\mathbf{r}$ planes, to be here denoted $\mathbf{Q}^{S_V}(z,\omega)$, say

$$\mathbf{Q}^{S_V}(z,\omega) = \int_V \mathbf{q}^{V}(\mathbf{r}')\delta(\boldsymbol{\omega}\cdot(\mathbf{r}-\mathbf{r}'))\,\mathrm{d}\mathbf{r}' = \int_{s_V(z,\omega)} \mathbf{q}^{V}(\mathbf{r}'\,|\,\boldsymbol{\omega}\cdot\mathbf{r}' = z)\,\mathrm{d}s(z,\omega) \tag{21}$$

By definition as well, the inverse Radon transform conversely yields $\mathbf{q}^{V}(\mathbf{r})$ as

$$\mathbf{q}^{V}(\mathbf{r}) = -\frac{1}{8\pi^2}\int_{\Omega} \mathbf{Q}^{S''_V}(z,\omega)\,\mathrm{d}\omega \tag{22}$$

with $\mathbf{Q}^{S''_V}(z,\omega)$ the second z-derivative of $\mathbf{Q}^{S_V}(z,\omega)$. When $\mathbf{q}^V(\mathbf{r}) = X_V(\mathbf{r})$, $\mathbf{Q}^{S_V}(z,\omega) = s_V(z,\omega)$, and Eq. (22) yields Eq. (12b). Similarly the expression given in Eq. (12a) of the v volume of some V bounded domain of R^3 in terms of its sections by planes corresponds to a more general relation between a pair of $\mathbf{h}(\mathbf{r})$ and $\mathbf{q}(\mathbf{r})$ summable functions over R^3 and their $\mathbf{H}(z,\omega)$, $\mathbf{Q}(z,\omega)$ Radon transforms, which is analogous to the Plancherel theorem writing

$$\int \mathbf{h}(\mathbf{r}) : \mathbf{q}(\mathbf{r})\,\mathrm{d}\mathbf{r} = -\frac{1}{8\pi^2}\int_\Omega \left(\int_{-\infty}^{+\infty} \mathbf{H}(z,\omega) : \mathbf{Q}''(z,\omega)\,\mathrm{d}z\right)\mathrm{d}\omega = -\frac{1}{8\pi^2}\int_\Omega \left(\int_{-\infty}^{+\infty} \mathbf{H}''(z,\omega) : \mathbf{Q}(z,\omega)\,\mathrm{d}z\right)\mathrm{d}\omega \tag{23}$$

or similarly for $\mathbf{h}^V(\mathbf{r})$ and $\mathbf{q}^V(\mathbf{r})$ functions defined over V. [8] When $\mathbf{h}^V(\mathbf{r}) = \mathbf{q}^V(\mathbf{r}) = X_V(\mathbf{r})$, Eq. (23) yields Eq. (12a) giving v. Furthermore, since the Radon transform $\mathrm{RT}[\mathbf{h}(\mathbf{r}),\mathbf{q}(\mathbf{r})]$ of a $(\mathbf{h},\mathbf{q})(\mathbf{r})$ convolution writes

$$\mathrm{RT}\left[\int \mathbf{h}(\mathbf{r}-\mathbf{r}') : \mathbf{q}(\mathbf{r}')\mathrm{d}\mathbf{r}'\right] = \int_{-\infty}^{+\infty} \mathbf{H}(z-z',\omega) : \mathbf{Q}(z',\omega)\,\mathrm{d}z' \tag{24}$$

it comes, from $\xi_V(\omega,\mathbf{r})$ in Eq. (8), that the inverse Radon transform of $\mathbf{\Gamma}(\mathbf{r}-\mathbf{r}')$ writes

$$\mathbf{\Gamma}(\mathbf{r}-\mathbf{r}') = -\frac{1}{8\pi^2}\int_\Omega \mathbf{t}^{P(\omega)}\delta''(z-z',\omega)\,\mathrm{d}\omega \tag{25}$$

Its integral over a V domain yields Eq. (20a) according to Eqs. (23) and (24), with $\mathbf{h}(\mathbf{r}-\mathbf{r}') = \mathbf{\Gamma}(\mathbf{r}-\mathbf{r}')$ and $\mathbf{q}(\mathbf{r}) = X_V(\mathbf{r})$. Removing $\mathbf{t}^{P(\omega)}$ from the right-hand side of Eq. (25) yields the plane wave expansion of the $\delta(\mathbf{r}-\mathbf{r}')$ function.

The Radon transform method provides a formally simple calculation of the $\mathbf{t}^V(\mathbf{r})$ modified Green operator integral related to a V general bounded domain shape, and of its $\overline{\mathbf{t}^V}$ average over V, based on a geometric characterisation of V. It is expected helpful in further manipulations of such operators which are essential in the resolution of the "inclusion" problem, and related ones, and this geometrical viewpoint may also help in differently seek for relevant resolution approximations, as next commented.

3.2. Application to the Eshelby inclusion problem

The now classical Eshelby problem of an isolated inclusion (or inhomogeneity) of $\mathbf{C}I$ elasticity moduli in an infinite medium of $\mathbf{C}$ moduli supporting the $\mathbf{E}$ strain applied from infinity ends to solve an integral equation of the form, with $\Delta\mathbf{C} = \mathbf{C}I - \mathbf{C}$ and $\boldsymbol{\varepsilon}^V(\mathbf{r})$ the strain field in V

$$\boldsymbol{\varepsilon}^V(\mathbf{r}) = \mathbf{E} - \int \Gamma(\mathbf{r}-\mathbf{r}') : \Delta\mathbf{C}X_V(\mathbf{r}') : \boldsymbol{\varepsilon}^V(\mathbf{r}')\,\mathrm{d}\mathbf{r}' \tag{26a}$$

In using $\Delta\mathbf{C}X_V(\mathbf{r}) : \boldsymbol{\varepsilon}^V(\mathbf{r}) = \Delta\mathbf{C} : \boldsymbol{\varepsilon}^V(\mathbf{r}) = \mathbf{q}^V(\mathbf{r}) : \mathbf{E} = \mathbf{p}^V(\mathbf{r})$, Eq. (26a) becomes

$$\mathbf{q}^V(\mathbf{r}) = \Delta\mathbf{C} : \left[\mathbf{I}X_V(\mathbf{r}) - \int_V \mathbf{\Gamma}(\mathbf{r}-\mathbf{r}') : \mathbf{q}^V(\mathbf{r}')\,\mathrm{d}\mathbf{r}'\right] \tag{26b}$$

Considering only the mean polarisation fields over V, which are enough to obtain the mean stress and strain fields in the V domain, Eq. (26b) yields, introducing $\mathbf{t}^V(\mathbf{r})$

$$\overline{\mathbf{q}^V} = \frac{1}{v}\int_V \mathbf{q}^V(\mathbf{r})\,\mathrm{d}r = \Delta\mathbf{C} : \left[\frac{\mathbf{I}}{v}\int_V X_V(\mathbf{r})\,\mathrm{d}\mathbf{r} - \frac{1}{v}\int_V \mathbf{t}^V(r) : \mathbf{q}^V(\mathbf{r})\,\mathrm{d}\mathbf{r}\right] \tag{26c}$$

[8] With a z-integral over the $2D_V(\omega)$ breadth of V then.

For problems where the polarisation field in V is uniform, $\mathbf{q}^V$ say, the classical solution simply writes

$$\mathbf{q}^V = [[\Delta\mathbf{C}]^{-1} + \overline{\mathbf{t}^V}]^{-1} = \mathbf{T}^V \tag{27}$$

in terms of the $\overline{\mathbf{t}^V}$ mean Green operator integral over V. The $\overline{\mathbf{t}^V}$ expression provided by Eq. (20b) can therefore be directly used in Eq. (27). This type of solution is also generally used for non uniform $\mathbf{q}^V(\mathbf{r})$ fields, under the so-called mean field approximation, which amounts to a priori replace $\mathbf{q}^V(\mathbf{r})$ by its mean value in the right hand side of Eq. (26c) (i.e. to approximate the $\overline{(\mathbf{\Gamma},\mathbf{q}^V)}^V$ average over V of the right-hand side $(\mathbf{\Gamma},\mathbf{q}^V)(\mathbf{r})$ convolution in Eq. (26b) by $\overline{\mathbf{t}^V} : \underline{\mathbf{q}^V}$), to directly write, instead of Eq. (26c)

$$\underline{\mathbf{q}^V} = \frac{1}{V}\int_V \mathbf{q}^V(\mathbf{r})\,\mathrm{d}\mathbf{r} = \Delta\mathbf{C} : \left[\frac{\mathbf{I}}{v}\int_V X_V(\mathbf{r})\,\mathrm{d}r - \left(\frac{1}{v}\int_V \mathbf{t}^V(\mathbf{r})\,\mathrm{d}\mathbf{r}\right) : \underline{\mathbf{q}^V}\right] = \Delta\mathbf{C} : \left[\mathbf{I} - \overline{\mathbf{t}^V} : \underline{\mathbf{q}^V}\right] \tag{28}$$

In Eq. (28), the underbar indicates that $\underline{\mathbf{q}^V}$ is an approximation, in general differing from the exact $\overline{\mathbf{q}^V}$ mean field solution. This $\underline{\mathbf{q}^V}$ solution yields the mean strain over V as $\underline{\boldsymbol{\varepsilon}^V} = [\Delta\mathbf{C}]^{-1} : \underline{\mathbf{q}^V} : \mathbf{E} = \underline{\mathbf{A}^V} : \mathbf{E}$, and from it the mean stress $\underline{\boldsymbol{\sigma}^V} = \mathbf{C}I : \underline{\boldsymbol{\varepsilon}^V}$, with

$$\underline{\mathbf{q}^V} = [[\Delta\mathbf{C}]^{-1} + \overline{\mathbf{t}^V}]^{-1} = \underline{\mathbf{T}^V} = \left(\int_\Omega \left[[\Delta\mathbf{C}]^{-1} + \mathbf{t}^{P(\omega)}\right]\overline{\psi_V}(\omega)\,\mathrm{d}\omega\right)^{-1} \tag{29a}$$

$$\underline{\mathbf{A}^V} = [\mathbf{I} + \overline{\mathbf{t}^V} : \Delta\mathbf{C}]^{-1} = \left(\int_\Omega \left[\mathbf{I} + \mathbf{t}^{P(\omega)} : \Delta\mathbf{C}\right]\overline{\psi_V}(\omega)\,\mathrm{d}\omega\right)^{-1} \tag{29b}$$

and similarly for the uniform field in ellipsoids, as a special case of uniform $\mathbf{q}^V = \mathbf{T}^V = \Delta\mathbf{C} : \mathbf{A}^V$ operators, when $\psi_V(\omega)$ is uniform.

The Radon transform method is in these cases of use to calculate $\overline{\mathbf{t}^V}$, for any type of V bounded domain, from geometrical considerations aiming to calculate (exactly or approximately) the $\overline{\psi_V}(\omega)$ mean weight function. As seen in part 2, if V is multiply connected (i.e. a group of inclusions), pair interaction terms are formally contained in $\overline{\psi_V}(\omega)$, this whether the relative pair positions of neighbouring inclusions in such a group are fully explicited or are treated in some statistical average. Therefore, as an alternative to separately consider V_i individual domains (of a given phase) and individually approximating them as ellipsoids to simplify calculations (which remain complicated if pair interactions are accounted for), a global "estimate" of $\overline{\psi_V}(\omega)$, for V taken as the union of the V_i set, can be relevant, according to the problem of concern.

In the general case represented by Eq. (26b), the resolution from the Radon transform method, using appropriately Eqs. (20)–(25), needs the inverse Radon transform of each term, to write

$$\mathbf{Q}^{s''_V}(z,\omega) = \Delta\mathbf{C} : \left(\mathbf{I}s''_V(z,\omega) - \frac{\partial^2}{\partial z^2}\left(\int_{D_V^-(\omega)}^{D_V^+(\omega)} \mathbf{P}(z-z',\omega) : \mathbf{Q}^{s_V}(z',\omega)\,\mathrm{d}z'\right)\right) \tag{30}$$

where the Radon transform of the $(\mathbf{\Gamma},\mathbf{q}^V)(\mathbf{r})$ convolution is written

$$\mathrm{RT}\left[\int \mathbf{\Gamma}(\mathbf{r}-\mathbf{r}') : \mathbf{q}^V(\mathbf{r}')\,\mathrm{d}\mathbf{r}'\right] = \int_{D_V^-(\omega)}^{D_V^+(\omega)} \mathbf{P}(z-z',\omega) : \mathbf{Q}^{s_V}(z',\omega)\,\mathrm{d}z' \tag{31}$$

$\mathbf{P}(z-z',\omega)$ stands for the Radon transform of $\mathbf{\Gamma}(\mathbf{r}-\mathbf{r}')$, the $\mathbf{P}''(z-z',\omega)$ second z-derivative of which is given by Eq. (25), and $\mathbf{Q}^{s_V}(z,\omega)$ for the one of $\mathbf{q}^V(\mathbf{r})$. This is formally allowed although the $\mathbf{\Gamma}(\mathbf{r}-\mathbf{r}')$ is not a locally summable function everywhere (due to its singularity at $\mathbf{r}'=\mathbf{r}$), in the sense of generalised functions, and where, furthermore, $\mathbf{\Gamma}(\mathbf{r}-\mathbf{r}')$ is to be understood as its "regularised form". This means the well known decomposition of $\mathbf{\Gamma}(\mathbf{r}-\mathbf{r}')$ into a delta function concentrated at $\mathbf{r}$, the local part, and the complementary non-local part, everywhere coinciding with $\mathbf{\Gamma}(\mathbf{r}-\mathbf{r}')$ except in an infinitesimal neighbourhood of $\mathbf{r}$, V^ε say, where it is zero. Typically, integrating Eq. (25) over the whole space, i.e., integrating over a sphere of infinite radius centred at $\mathbf{r}$, yields the modified Green operator integral given by $\mathbf{t}^V(\mathbf{r})$ in Eq. (2) when V is a sphere, $\mathbf{t}^S$ say. Being invariant with respect to the sphere size, this operator also corresponds to an infinitesimal $V^{S\varepsilon}$ sphere around $\mathbf{r}$, and the local part of $\boldsymbol{\Gamma}(\mathbf{r}-\mathbf{r}')$ therefore is $\mathbf{t}^S\delta(\mathbf{r}-\mathbf{r}')$, while the $\boldsymbol{\Gamma}^S(\mathbf{r}-\mathbf{r}')$ non-local operator part has a zero integral over the whole space. This regularised form of $\boldsymbol{\Gamma}(\mathbf{r}-\mathbf{r}')$ is not unique since the $\mathbf{r}$ point can be isolated by an infinitesimal surrounding volume of any V shape, in which case $\boldsymbol{\Gamma}(\mathbf{r}-\mathbf{r}') = \mathbf{t}^V(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}') + \boldsymbol{\Gamma}^V(\mathbf{r}-\mathbf{r}')$. The $\boldsymbol{\Gamma}^V(\mathbf{r}-\mathbf{r}')$ integral over the whole space is not zero then, and it is uniform when V is ellipsoidal. Taking $\boldsymbol{\chi} = \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$, and $I_{V\varepsilon}$ equal to zero inside V^ε (i.e. for $\mathbf{r}-\mathbf{r}' < \varepsilon^V(\boldsymbol{\chi})$) and 1 if not, this decomposition writes

$$\boldsymbol{\Gamma}(\mathbf{r}-\mathbf{r}') = \mathbf{t}^V(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}') + I_{V\varepsilon}\boldsymbol{\Gamma}(\mathbf{r}-\mathbf{r}')$$

Note that the usual decomposed form of $\boldsymbol{\Gamma}(\mathbf{r}-\mathbf{r}')$, written $\mathbf{E}^V(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}') + \mathbf{F}^V(\boldsymbol{\chi})\frac{1}{|\mathbf{r}-\mathbf{r}'|^3}$ is retrieved when Eq. (25) is reformulated as

$$\boldsymbol{\Gamma}(\mathbf{r}-\mathbf{r}') = -\frac{1}{8\pi^2}\int_\Omega \mathbf{t}^{P(\omega)}\delta''(\omega\cdot(\mathbf{r}-\mathbf{r}'))\,\mathrm{d}\omega = -\frac{1}{4\pi}\left(\int_\Omega \frac{\mathbf{t}^{P(\omega)}}{2\pi}\frac{2\delta(\omega\cdot\chi)}{(\omega\cdot\chi)^2}\,\mathrm{d}\omega\right)\frac{1}{|\mathbf{r}-\mathbf{r}'|^3}$$

the Ω integral within brackets reducing to an integral over a $C(\boldsymbol{\chi})$ unit circle around the $\boldsymbol{\chi}$-direction, since non-zero terms are those for which $\omega\cdot\chi = 0$.

Therefore, in any choice of V^ε, the expression of $\boldsymbol{\Gamma}(\mathbf{r}-\mathbf{r}')$ from Eq. (25) does account for both the local and the non-local part. However, this remarkably simple form of $\boldsymbol{\Gamma}(\mathbf{r}-\mathbf{r}')$, which simply allows to explicit the $\mathbf{t}^V(\mathbf{r})$ operator provided the knowledge of appropriate geometrical characteristics of V, does not allow the economy of separating these local and non-local parts for treating the inclusion problem(s) more precisely than within the mean field approximation above recalled.

Entering these developments here would drive us too far from our main topic, which is the (exact or approximate) determination of $\overline{\psi_V}(\omega)$ from geometrical consideration. Further discussion of the resolution of the inclusion problem from the Radon transform method will therefore be delayed to a separate presentation.

Remaining in the mean field approximation context, a few additional comments are reported in the next part, with regard to the use of the Radon transform expression of the modified Green operator integrals in simple cases of effective moduli estimates. Let us call $\overline{\mathbf{q}^V} = \overline{\mathbf{T}^V}$ the exact solution of the inclusion problem, differing from the $\underline{\mathbf{T}^V}$ mean field approximation unless for single V ellipsoids where $\overline{\mathbf{T}^V} = \underline{\mathbf{T}^V} = \mathbf{T}^V$.

3.3. Application to effective elasticity moduli estimates for inclusion reinforced matrices

There are many different approaches of this effective moduli estimate problem for heterogeneous materials, in general making use of approximations in the resolution procedure, and we cannot either enter this difficult question here. We just aim to point out that the Radon transform method can be an interesting new way to approach these problems, with regard to inclusion shapes on the one hand, and also with regard to the characteristic shapes of their spatial pair position distributions. We limit our discussion to two-phase inclusion matrix structures.

For a f volume fraction of V_i inclusions of all same $\mathbf{C}I$ moduli (not necessarily shape-identical at this point) in the infinite $\mathbf{C}$ medium, Eq. (26a) is generally said to hold, with an integral over a RV "representative material volume", if taking the modified Green operator to act on $(\mathbf{p}^V(\mathbf{r}) - \overline{\mathbf{p}^*})$ (Walpole, 1981; Willis, 1981), where $\overline{\mathbf{p}^*}$ is the volume average of $\mathbf{p}^V(\mathbf{r})$ over RV, and a priori taking for V all the V_i inclusions in RV. Under this condition, the $\mathbf{E}$ strain is replaced by $\mathbf{E}o(\mathbf{r}) = \mathbf{E} + (\int_{RV}\boldsymbol{\Gamma}(\mathbf{r}-\mathbf{r}')\,\mathrm{d}\mathbf{r}') : \overline{\mathbf{p}^*} = \mathbf{E} + \mathbf{t}^{RV}(\mathbf{r}) : \overline{\mathbf{p}^*}$. In denoting $\overline{\mathbf{E}o}$ the volume average of $\mathbf{E}o(\mathbf{r})$ over V, and setting then $\mathbf{p}^V(\mathbf{r}) = \mathbf{q}o^V(\mathbf{r}) : \overline{\mathbf{E}o}$ together with $\overline{\mathbf{p}^*} = \overline{\mathbf{q}o^*} : \overline{\mathbf{E}o}$, provides the relations (for $\mathbf{r} \in V \subset RV$)

$$\overline{\mathbf{E}o} = [\mathbf{I} - \overline{(\mathbf{t}^{RV})}^V : \overline{\mathbf{q}o^*}]^{-1} : \mathbf{E}, \quad \mathbf{E}o(\mathbf{r}) = (\mathbf{I} + \Delta\mathbf{t}_V^{RV}(\mathbf{r}) : qo^*) : \overline{\mathbf{E}o} \tag{32a}$$

with

$$\overline{(\mathbf{t}^{RV})}^V = \frac{1}{V}\int_V \mathbf{t}^{RV}(\mathbf{r})\,\mathrm{d}\mathbf{r}, \quad \Delta\mathbf{t}_V^{RV}(\mathbf{r}) = \mathbf{t}^{RV}(\mathbf{r}) - \overline{(\mathbf{t}^{RV})}^V \tag{32b}$$

Solving Eq. (26a), where $\mathbf{E}o(\mathbf{r})$ replaces $\mathbf{E}$, with respect to the Radon transform of $\mathbf{q}o^V(\mathbf{r})$, yields

$$\mathbf{Q}o^{S''_V}(z,\omega) = \Delta\mathbf{C} : (\mathbf{I}s''_V(z,\omega) + \mathbf{A}^{s''_V}(z,\omega) : \overline{\mathbf{q}o^*}) - \Delta\mathbf{C} : \frac{\partial^2}{\partial z^2}\left(\int_{D_V^-(\omega)}^{D_V^+(\omega)} \mathbf{P}(z-z',\omega) : \mathbf{Q}o^{s_V}(z',\omega)\,\mathrm{d}z'\right) \tag{33}$$

instead of Eq. (30), with $\mathbf{A}^{S''_V}(z,\omega)$ standing for the second z-derivative of the Radon transform of $\Delta\mathbf{t}_V^{RV}(\mathbf{r})$

$$\mathbf{A}^{s''_V}(z,\omega) = \mathbf{t}^{P(\omega)}\left(s''_{RV}(z,\omega) - \frac{1}{v}\int_{D_V^-(\omega)}^{D_V^+(\omega)} s''_{RV}(z',\omega)s_V(z',\omega)\,\mathrm{d}z'\right) = \mathbf{t}^{P(\omega)}\left(\xi_{RV}(\omega,\mathbf{r}) - \overline{(\xi_{RV})}^V(\omega)\right) \tag{34}$$

Denoting from Eq. (34) that $\frac{1}{v}\int_{D_V^-(\omega)}^{D_V^+(\omega)} \mathbf{A}^{S''_V}(z,\omega)s_V(z,\omega)\,\mathrm{d}z = \mathbf{0}\ \forall\boldsymbol{\omega}$, averaging over V will provide an exact mean $\overline{\mathbf{q}o^V}$ value expression identical to the exact $\overline{\mathbf{q}^V}$ solution related to the isolated V domain. It will therefore formally come for $\overline{\mathbf{q}^V}$ now related to a f volume fraction of V domains, or of V_i inclusions, in the matrix, and for the related $\overline{\mathbf{C}^{\mathrm{eff}}}$ material effective moduli

$$\overline{\mathbf{p}^V} = \overline{\mathbf{T}^V} : \overline{\mathbf{E}o} = \overline{\mathbf{T}^V} : \left[\mathbf{I} - \overline{(\mathbf{t}^{RV})}^V : \overline{\mathbf{q}o^*}\right]^{-1} : \mathbf{E} = \overline{\mathbf{q}^V} : \mathbf{E}, \quad \overline{\mathbf{C}^{\mathrm{eff}}} = \mathbf{C} + f\overline{\mathbf{q}^V} \tag{35}$$

As for the isolated inclusion problem, an *approximate* mean $\underline{\mathbf{q}o^V}$ solution can be obtained in approximating the $\overline{(\boldsymbol{\Gamma},\mathbf{q}o^V)}^V$ average of the $(\boldsymbol{\Gamma},\mathbf{q}o^V)(\mathbf{r})$ convolution by $\overline{\mathbf{t}^V} : \underline{\mathbf{q}o^V}$. According to what preceeds, $\underline{\mathbf{q}o^V} = \underline{\mathbf{T}^V}$ defined from Eq. (29a), what thus yields the $\underline{\mathbf{q}^V}$ and $\underline{\mathbf{C}^{\mathrm{eff}}}$ approximate solution

$$\underline{\mathbf{p}^V} = \underline{\mathbf{T}^V} : \underline{\mathbf{E}o} = \underline{\mathbf{T}^V} : \left[\mathbf{I} - \overline{(\mathbf{t}^{RV})}^V : \underline{\mathbf{q}o^*}\right]^{-1} : \mathbf{E} = \underline{\mathbf{q}^V} : \mathbf{E};\ \underline{\mathbf{C}^{\mathrm{eff}}} = \mathbf{C} + f\underline{\mathbf{q}^V} \tag{36}$$

with the corresponding $\underline{\mathbf{q}o^*}$ (and thus $\underline{\mathbf{E}o}$) average fields. Since $\mathbf{q}o^V(\mathbf{r})$ is zero out of V, its average over RV, as introduced in Eqs. (32), must write

$$\overline{\mathbf{q}o^*} = \frac{f}{V}\int_V \mathbf{q}o^V(\mathbf{r})\,\mathrm{d}\mathbf{r} = f\overline{\mathbf{q}o^V} = f\overline{\mathbf{T}^V} \tag{37a}$$

$$\underline{\mathbf{q}o^*} = \frac{f}{V}\int_V \mathbf{q}o^V(\mathbf{r})\,\mathrm{d}\mathbf{r} = f\underline{\mathbf{q}o^V} = f\underline{\mathbf{T}^V} \tag{37b}$$

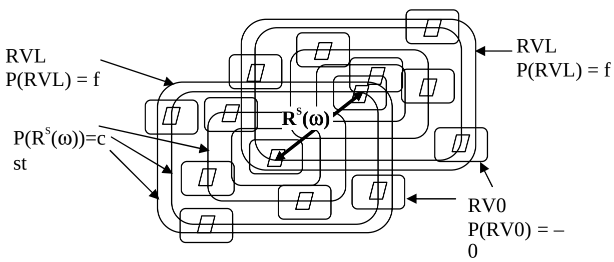

Fig. 5. A RV representative volume (drawn around two inclusions) as a limit position correlation length, and homothetic $P(R^S(\omega)) = \text{cst}$ surfaces, for a uniform distribution of V_i identical inclusions.

when, respectively, considering the exact solution and the mean field approximation. As set, the only difference between approximate and exact resolutions is expected to come from the difference of $\overline{\mathbf{T}^V}$ from $\underline{\mathbf{T}^V}$.

Now, both the $\overline{(\mathbf{t}^{RV})^V}$ and $\overline{\mathbf{T}^V}$ (respectively, $\underline{\mathbf{T}^V}$) operators contributing in $\overline{\mathbf{q}^V}$ (respectively, in $\underline{\mathbf{q}^V}$), a priori depend on the relative pair positions of the V_i inclusions, both through V and through RV. V can be a particular pattern or a statistically specified inclusion arrangement. The S surfaces of section 2 can be seen as representing surfaces of constant conditional probability $P(R^S(\omega))$ to find an inclusion centre at a $R^S(\omega)$ distance from some inclusion centre taken as origin (Ponte-Castaneda and Willis, 1995; Bornert et al., 1996). In this case RV can be seen as the characteristic shape of a uniform position pair distribution for the V_i inclusions, as simplistically exemplified in Fig. 5, for all identical inclusions: around any inclusion, from statistical homogeneity, RV specifies homothetic S surfaces such that $P = 0 \forall R(\boldsymbol{\omega}) \leqslant R^{S_0}(\boldsymbol{\omega})$, and $P = f \, \forall R(\boldsymbol{\omega}) \geqslant R^{SL}(\boldsymbol{\omega})$. Space filling is only realisable then, when the V_i inclusions are congruent and congruent to RV as well. [9] We limit ourselves to the following comments about $\overline{(\mathbf{t}^{RV})^V}$ in the context of the mean field approximation for which $\underline{\mathbf{T}^V}$ has been shown accessible to a calculation from the mean weight function characteristic of V.

For general RV shapes, the inverse Radon transform of $\overline{(\mathbf{t}^{RV})^V}$ yields

$$\overline{(\mathbf{t}^{RV})^V} = \int_\Omega \mathbf{t}^{P(\omega)} \left(\frac{1}{V} \int_{D_V^-}^{D_V^+} \psi_{RV}(\omega, \mathbf{r}) s_V(z,\omega)\,\mathrm{d}z \right) \mathrm{d}\omega = \int_\Omega \mathbf{t}^{P(\omega)} \overline{(\psi_{RV})^V}(\omega)\,\mathrm{d}\omega \tag{38}$$

As $\overline{\mathbf{t}^V}$, $\overline{(\mathbf{t}^{RV})^V}$ takes the form of a weighted average of the elementary $\mathbf{t}^{P(\omega)}$ operators. But the weight function of the RV domain is involved in terms of its $\overline{(\psi_{RV})^V}(\omega)$ average over its V sub-domain, unless for ellipsoidal RV shapes, for which one has, $\forall V \subset RV$ and $\forall \mathbf{r} \in V$, $\psi_{RV}(\omega, \mathbf{r}) = \psi_{RV}(\omega)$, and $\overline{(\mathbf{t}^{RV})^V} = \mathbf{t}^{RV} = \mathbf{t}^{RV}(\mathbf{r})$ (thus $\mathbf{E}o(\mathbf{r}) = \mathbf{E}o$). Otherwise, $\overline{(\psi_{RV})^V}(\omega)$ remains dependent on the spatial arrangement of the V_i inclusions that constitutes V in RV, either for a given pattern, or for a statistically defined position distribution. However, since by definition $RV \supset V \gg V_i$, one can see RV as nearly the convex hull of $V \gg V_i$, such that $D_V(\omega) \approx D_{RV}(\omega))$ in Eq. (38) while, in average over all realisations of a same RV inclusion position distribution, statistical homogeneity allows to take $s_V(z,\omega) \approx f s_{RV}(z,\omega)$. This would give $\overline{(\psi_{RV})^V}(\omega) \approx \overline{\psi_{RV}}(\omega)$ and $\overline{(\mathbf{t}^{RV})^V} \approx \overline{\mathbf{t}^{RV}}$, i.e. a f and V-independent "global distribution effect". Approximations of $\overline{\mathbf{t}^{RV}}$ can be thought off in the same way as for $\overline{\mathbf{t}^V}$, i.e. from approximations of the $\overline{\psi_{RV}}(\omega)$ mean weight function.

The approximate $\underline{\mathbf{q}^V}$ solution finally writes

$$\underline{\mathbf{q}^{\mathrm{V}}} = \underline{\mathbf{T}^V} : \left[\mathbf{I} - f \overline{(\mathbf{t}^{RV})^V} : \underline{\mathbf{T}^V} \right]^{-1} = \left[(1-f)[\underline{\mathbf{T}^V}]^{-1} + f\left([\Delta\mathbf{C}]^{-1} + \overline{\mathbf{t}^V} - \overline{(\mathbf{t}^{RV})^V} \right) \right]^{-1} \tag{39a}$$

$$\approx \left[(1-f)[\underline{\mathbf{T}^V}]^{-1} + f\left([\Delta\mathbf{C}]^{-1} + \overline{\mathbf{t}^V} - \overline{\mathbf{t}^{RV}} \right) \right]^{-1} \tag{39b}$$

Involving two mean operators which can be geometrically calculated.

3.4. Concluding remarks on the use of the Radon transform in inclusion problems

The Radon transform method proves of interest at least in inclusion problems resulting in (or reduced to) the calculation of mean operators related to particular inclusion or domain shapes when these operators can be decomposed, as the mean Green operator integrals for these domains, as a weighted average over an angular distribution of elementary operators. Extended usefulness in broader contexts will be presented in

[9] Together with a fractal size distribution of the inclusions down to infinitesimal. Otherwise $f_{\max} < 1$.

forthcoming work. As a first step, the present paper mainly aims to stress that, as far as such mean operators are sufficient in the problem of concern, they can be obtained from the (exact or approximate) calculation of the $\overline{\psi_V}(\omega)$ mean weight function(s) characteristic of the involved V domain(s), what can be performed from geometrical considerations. Section 4 exemplifies such mean weight function calculations for simply connected, convex centrosymmetric, non-ellipsoidal domains, and discusses relevant approximations, when necessary, in comparison to a "best ellipsoidal fitting" of the domain shapes.

4. Mean weight function approximations for non-ellipsoidal inclusions

The exact $\psi_V(\omega)$ weight function for V ellipsoids is analytically given from the $D_V(\omega)$ support function as $(R^3_{V^*}(\omega)/3v^*)$, with $R_{V^*}(\omega) = 1/D_V(\omega)$. Exact calculations of the mean $\overline{\psi_V}(\omega)$ weight function (or of the $\psi_V(\omega, \mathbf{r})$ function field in V) for general V inclusions can be more or less tediously obtained when the inclusion shape is analytically specified by a $P(\mathbf{r}) = 0$ function, or under parametric forms. Planar section areas of such domains in all spatial directions, and then the required derivatives, can be at least numerically, but also analytically in certain cases, determined such as to precisely obtain $\overline{\psi_V}(\omega)$. They can also be obtained from 3D analyses of experimental data, from X-ray tomography for example. The ellipsoidal approximation of non-ellipsoidal V inclusion shapes that is introduced in inclusions or homogenisation problems for simpler analytical resolutions, corresponds to approximate the exact $\overline{\psi_V}(\omega)$ mean weight function of V, by the weight function of the "closest" Ve ellipsoidal shape (from a least square method for example), say taking $\overline{\psi_V}(\omega) \approx \psi_{\mathrm{Ve}}(\omega) = R^3_{\mathrm{Ve}^*}(\omega)/3ve^*$. For inclusion shapes such as the ones illustrated in Fig. 1a and b the best ellipsoidal fit remains a rough approximation, and better approximations of $\overline{\psi_V}(\omega)$ can be figured out. In the particular class of convex, not necessarily centrosymmetric, V domains, an expected better approximation in many cases could be to use the support function of V, to approximate $\overline{\psi_V}(\omega)$ as $\approx \tilde{\psi}_V(\omega) = 1/(3v^* D^3_V(\omega)) = R^3_{V^*}(\omega)/(3v^*)$ (in reference to the V^* reciprocal body of V, and with $\int_\Omega (R^3_{V^*}(\omega)/3v^*)\,\mathrm{d}\omega = 1$). Considering for example both the cube and the octahedron exemplified in Fig. 3, and also all the intermediate cuboidal shapes of equation $\sum_{i=1}^{3}(x_i)^{2n} = 1$, with 0.5 (octahedron) $\leqslant n \leqslant \infty$ (cube), the sphere ($n = 1$) would be the "best" ellipsoidal approximation for all of them. Both the exact $\overline{\psi_V}(\omega)$ mean weight function and its $\tilde{\psi}_V(\omega)$ "support function approximation" can be calculated for the cube and the octahedron, respectively from analytical expressions of all their section areas for $\overline{\psi_V}(\omega)$ (see Appendix B), and from their support functions for $\tilde{\psi}_V(\omega)$, which respectively write $D(\omega) = |\omega_1| + |\omega_2| + |\omega_3|$, and $D(\omega) = \max(|\omega_1|, |\omega_2|, |\omega_3|)$, in the $\boldsymbol{\omega}$-direction. Since they are the oppositely farthest convex centrosymmetric shapes from the sphere, the exact $\overline{\psi_V}(\omega)$ mean weight functions of these shapes are expected to exhibit the largest difference with their related $\tilde{\psi}_V(\omega)$ approximation. These exact and approximate weight function for the cube and the octahedron are compared in Fig. 6, which shows that, although the support function approximation provides a smoother weight function than the exact mean one, it is obviously much better than using the weight function of the "best fitting" ellipsoid, which in the present case is the sphere.

The comparison of the corresponding exact $\overline{\mathbf{t}^V}$ and approximate $\tilde{\mathbf{t}}^V$ calculations of the mean modified Green operator integral is performed in the case of isotropic elasticity. In this case, from Eq. (18a), the $\mathbf{t}^{P(\omega)}$ elementary operators share into two $a\boldsymbol{\delta}^{a(\omega)} + b\boldsymbol{\delta}^{b(\omega)}$ parts, related to the two independent ($a = (1-2\nu)/[2\mu(1-\nu)]$, $b = 1/4\mu$) constant terms. These two parts can be integrated separately from Eq. (20b) as $\overline{\boldsymbol{\delta}^{aV}} = \int_\Omega \boldsymbol{\delta}^{a(\omega)}\overline{\psi_V}(\omega)\,\mathrm{d}\omega$, (respectively, $\overline{\boldsymbol{\delta}^{bV}}$), for $\overline{\mathbf{t}^V} = a\overline{\boldsymbol{\delta}^{aV}} + b\overline{\boldsymbol{\delta}^{bV}}$, or as $\tilde{\boldsymbol{\delta}}^{aV} = \int_\Omega \boldsymbol{\delta}^{a(\omega)}\tilde{\psi}_V(\omega)\,\mathrm{d}\omega$, (respectively, $\tilde{\boldsymbol{\delta}}^{bV}$), for $\tilde{t}^V = a\tilde{\boldsymbol{\delta}}^{aV} + b\tilde{\boldsymbol{\delta}}^{bV}$, regardless of the elasticity moduli values in (a, b). In the main symmetry axes of the considered shapes, all these operators reduce to three $(iiii), (iijj), (ijij)$ components and, noticing that $\delta^{a(\omega)}_{iijj} = \delta^{a(\omega)}_{ijij}$, both $\overline{t^V}$ and $\tilde{\mathbf{t}}^V$ involve only five different terms here calculated for the cube and the octahedron. For the n-cuboidal shapes, they have also been calculated for $\tilde{\mathbf{t}}^V$, but not for $\overline{\mathbf{t}^V}$ since only their

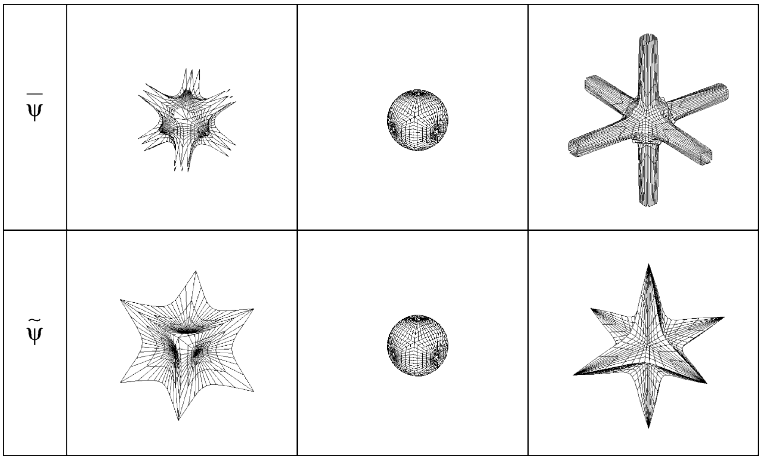

Fig. 6. $\overline{\psi_V}(\omega)$ exact and $\tilde{\psi}_V(\omega)$ approximate mean weight functions for the octahedron and the cube, at same scale (the conical branches of $\overline{\psi_V}(\omega)$ are truncated).

$\tilde{\psi}_V(\omega)$ approximate weight function is easily at analytical hand, [10] from their support function which writes

$$D^{(n)}(\omega) = \left(\sum_{i=1}^{3} \omega i^{2n/(2n-1)} \right)^{(2n-1)/2n} \tag{40}$$

The so-obtained five terms of $\tilde{\mathbf{t}}^V$ are plotted in Fig. 7 with respect to $N = (2n-1)/2n$ which ranges from 0 (octahedron) to 1 (cube). For these limit cases, as well as for the sphere ($N = 0.5$), the five different terms of $\overline{\mathbf{t}^V}$ are also plotted. Since the $\tilde{\mathbf{t}}^V$ approximation of $\overline{\mathbf{t}^V}$ is, for this family of shapes, the worst for the limit cases of the cube and the octahedron, the Fig. 7 shows that this approximation is good for both limit shapes and consequently it can be expected good as well for all intermediate cuboidal ones. The $\tilde{\mathbf{t}}^V$ approximation of $\overline{\mathbf{t}^V}$ is obviously better than the "best ellipsoidal" approximation, which here is the sphere $\mathbf{t}^S$ uniform operator ($\mathbf{t}^S_{iiii} - \mathbf{t}^S_{iijj} = 2\mathbf{t}^S_{ijij}$).

It is thus shown that the approximate calculation of the cuboids mean weight function from their support function is close enough to the exact one in the whole range between the octahedron and the cube, to provide a good approximate of the mean Green operator integral over inclusions of such shapes. This can be used to approximate all operators also having an inverse Radon transform on that same shape-characteristic mean weight function.

The exemplified case of cuboidal inclusions extends to more general cuboids by the same linear transform as for the sphere into an ellipsoid. All these non-ellipsoidal convex centrosymmetric shapes already allow to cover a much larger range of particle or void shapes that can be encountered in real materials, than ellipsoidal shapes only. For example, cuboidal inclusions of equation $\sum_{i=1}^{3} (\frac{x_i}{a_i})^{2n} = 1$ are characteristic of γ' precipitates in various γ/γ' superalloys (Estevez et al., 1995), and different γ/γ' alloys exhibit cuboids of various (n) smoothness or (a_i) anisotropic main dimensions. Their volume fraction is generally larger than the maximal allowed concentration of size-identical "best fitting" ellipsoids, what prohibits the ellipsoidal approximations, unless admitting an irrelevant fractal size distribution of precipitates.

[10] The equation of the intercept of a n-cuboid V with a $\lambda(\omega)$ plane of $\boldsymbol{\omega}$ normal unit vector at the distance $\lambda(\omega)$ from the origin i.e. $\omega_i x_i = \lambda(\omega)$, writes $\frac{\lambda}{\omega_3} = \frac{\omega_1 x_1}{\omega_3} + \frac{\omega_2 x_2}{\omega_3} - (1 - x_1^{2n} - x_2^{2n})^{\frac{1}{2n}}$, and its area has no (or no simple) analytical expression. But the $D^V(\omega)$ breadth is given by $\lambda(\omega)_{\max}$, solving the conditions $\partial\lambda/\partial x_i = 0$ for $i = 1, 2$, what provides the support function of Eq. (40).

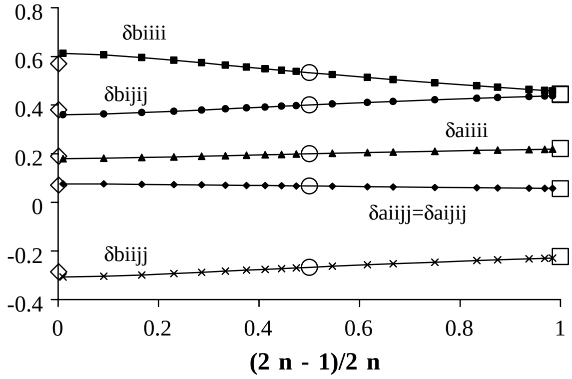

Fig. 7. The δ^{aV}_{ijkl} and δ^{bV}_{ijkl} different components of $\tilde{\mathbf{t}}^V$, from $\tilde{\psi}_V(\omega)$ for n-cuboids (lines), and of $\overline{\mathbf{t}^V}$ from $\overline{\psi_V}(\omega)$ for the octahedron, the sphere and the cube (open symbols).

From the given cuboidal examples, it likely holds for many convex, non-necessarily centrosymmetric, domains, that the $\tilde{\psi}_V(\omega)$ mean weight function approximation from the support function of the inclusion captures the main inclusion shape effect on the modified Green operator integral (and on related operators), and that this approximation is more accurate than an ellipsoidal approximation of the shape. One additional advantage of using the support function of a V domain (when convex) to approximate $\overline{\psi_V}(\omega)$ by $\tilde{\psi}_V(\omega)$ is that non-analytically defined convex domains which are defined from sets of analytically defined ones, such as being for example the convex hull, or the dilatation, of several domains, still have simply defined support functions. [11] This is only an example of approximation, of use for an important shape set. If not precise enough, or not valid, for particular convex inclusions or for non-convex ones, or for more general V bounded domains (inclusion patterns), approximations of the mean weight function can still be considered in appropriate manner with regard to the geometry, in order to both preserve the possibility of analytical calculations and a realistic description of the inclusion shapes and arrangements of concern. Such approximations concerning the mean weight function characteristics of general bounded domain shapes (from the domain support function or from any other more appropriate geometrical considerations), are expected to improve the behaviour estimate of matrices reinforced with inclusions (or weakened by voids) far from ellipsoids, since the inclusion shape acts at first order of the inclusion volume fractions in homogenisation calculations. This constitutes a first order improvement in many modelling contexts. An ellipsoidal approximation of a non-ellipsoidal inclusion position distribution symmetry is more acceptable since it acts at second order of the inclusion volume fractions. But in the superalloy example, position distributions of cuboidal precipitates are closer to an inclusion shape homothetic distribution than to an ellipsoidal one. Although the cases of inclusion spatial position distributions of non-ellipsoidal symmetry still deserve specific care, approximations of mean weight functions related to such distribution shape characteristics are also expected of interest to approximate operators that would depend on such functions. An other such situation of importance is the aggregate structure, the grains of which are polyhedra rather than ellipsoids, and the realisation of which is only safely consistent (in terms of two-point statistics) with congruent grains in uniformly grain-shape-homothetic position distribution, as established for ellipsoids in Bornert et al. (1996). The Radon transform method is likely also helpful in other inclusion-related contexts,

[11] The ω-breadth of the convex hull of several convex bodies is the maximum of their ω-breadths, the ω-breadth of the dilatation of several bodies is their ω-breadth sum. For the intercept of several bodies, one can refer to the reciprocal body which is the convex hull of the reciprocal bodies, etc.

for example in the very many fields involving evolutive inclusion boundary problems (Sabar et al., 1991), as well as when co-connected morphologies are concerned, since, as stressed in Section 2, it can also apply to unbounded domains.

5. Conclusion

The modified Green operator integral related to a general V bounded domain inside a matrix, has been expressed, for linear elasticity, in terms of its inverse Radon transform, as a weighted average, at each $\mathbf{r}$ point inside V, over an angular distribution of a single elementary operator. The weight function, geometrically defined from all the planar section areas of V passing through the $\mathbf{r}$ point, expresses the inverse Radon transform of the characteristic function of V. The elementary operator, referring to all possible plane orientations, or directions, in space, identifies to the platelet operator of same planar, or small axis, orientation. So re-interpreted in a geometrically more meaningful approach, the Green operator integral calculation can be simplified for general bounded domain shapes, and further manipulations may be simplified as well. The Radon transform method provides an alternative analytical resolution of the general inclusion problem in an infinite medium of general elasticity anisotropy, the inclusion being any bounded domain, single inclusion or inclusion patterns. It also provides insights for analytical resolutions of inclusion-related problems, such as effective moduli estimates of heterogeneous materials. The microstructural characteristics of importance come out to be weight functions related to the involved characteristic (inclusion or position distribution) shapes, and at first their mean value over the domains defined by these shapes. The Radon transform approach especially deserves consideration in inclusion-related problems that do not reduce to ellipsoidal inclusion geometries or ellipsoidal spatial arrangement symmetries. Resolution then can be thought of in terms of, geometrically-based, calculations of the mean weight functions of the involved characteristic shapes. In many circumstances, an approximation of the mean weight functions, if necessary, is likely to provide a more accurate final result than a best ellipsoidal fitting of the shapes themselves.

Appendix A. Identity of the elementary and platelet (or laminate) operators

The platelet operator expression in the platelet frame (with non-zero terms as given in Table 1) is directly obtained for the laminate structure for which, taking $z = x3$ as the layers normal, one has the localisation relations, between local ($\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$) and macroscopic ($\boldsymbol{\Sigma}$, $\mathbf{E}$) stresses and strains

$$\varepsilon_{ij} = E_{ij} \text{ for } i \text{ and } j \text{ in } (1,2); \quad \sigma_{i3} = \Sigma_{i3} \text{ for } i \text{ in } (1,3) \tag{A.1}$$

From the Eshelby isolated inclusion problem, the general strain expression in an ellipsoidal inclusion uniformly writes $\boldsymbol{\varepsilon} = \mathbf{E} + \mathbf{t} : \mathbf{p}$, with $\mathbf{p}$ a "polarisation stress", and $\mathbf{t}$ the ellipsoid operator. If the inclusion is of platelet (or laminate) shape, it is straightforward that from this strain expression, $t_{ijkl} = 0$ for all terms such that (i and j) and (k and l) are in (1,2), yielding for $\mathbf{t}$, the same remaining non-zero terms as given by $\Delta\mathbf{t}$ in Table 1 for the $z \equiv 3$ platelet. To next show that the non-zero sub-matrix of $\mathbf{t}$ for the z-laminate inclusion identifies to $\mathbf{N} = \mathbf{M}^{-1}$ of Table 1, one can refer to the dual Green "stress" approach (Zeller and Dederich, 1973), which involves dual $\mathbf{t}' = \mathbf{C} - \mathbf{C} : \mathbf{t} : \mathbf{C}$ operators (conversely $\mathbf{t} = \mathbf{S} - \mathbf{S} : \mathbf{t}' : \mathbf{S}$, with $\mathbf{S} = \mathbf{C}^{-1}$) from which one can write for the stresses, as for the strains, the dual relation $\boldsymbol{\sigma} = \boldsymbol{\Sigma} + \mathbf{t}' : \mathbf{e}$. Whatever the "polarisation strain $\mathbf{e}$" is in this stress expression, according to the relations given in Eq. (A.1), the $\mathbf{t}'$ operator for the z-laminate inclusion is therefore zero for all t'_{ijkl} terms with either i or j and either k or l equal to 3. It is in particular zero for the $(ijkl)$ indices corresponding to the non-zero $\mathbf{t}$ terms, i.e. those of the $\Delta\mathbf{t}'$ block corresponding to the $\Delta\mathbf{t}$ block of Table 1. For these terms, the $\mathbf{t}'$ expression versus $\mathbf{t}$ only involves $\Delta\mathbf{t}$ and the part of $\mathbf{C}$ given by the $\mathbf{M}$ matrix, and writes $\Delta\mathbf{t}' = \mathbf{0} = \mathbf{M} - \mathbf{M} : \Delta\mathbf{t} : \mathbf{M}$, yielding $\Delta\mathbf{t} = [\mathbf{M}]^{-1} = \mathbf{N}$, as in Section 2 from the Green/Fourier approach.

Appendix B. Calculation of the mean weight function of the cube and the octahedron

We consider the sections of the cube $(\max(|x_1|, |x_2|, |x_3|) = 1)$, and of the octahedron $(|x_1| + |x_2| + |x_3| = 1)$ with the plane $\omega_i x_i = \lambda$, where $\omega_1, \omega_2, \omega_3$ are the components of the $\boldsymbol{\omega}$ unit normal vector to the plane and λ the distance between the plane and the origin. The area of the section is determined first in the standard stereographic triangle $\omega_1 < \omega_2 < \omega_3$ and extended to the first octant by symmetry and permutations, and then to all directions by appropriate symmetries. The sections are polygon from three to six edges, and the section area S is determined from the area s of the section projection on the $x_3 = 0$ plane by the relation $S = s/\omega_3$. The area of the section projection is determined from the projection of the n section corners which are the intersections of the (λ) plane with the faces of the polyhedron, by the relation $s = \frac{1}{2}\sum_{i=1}^{n}(\omega_1^i \omega_2^{i+1} - \omega_1^{i+1}\omega_2^i)$, with the convention $\omega_j^{n+1} = \omega_j^1$, $j = 1, 2$. The area S is a polynomial function of λ, easy to derive to obtain the mean $\overline{\psi_V}(\omega)$ weight function from Eq. (20b). So doing for the cube and the octahedron respectively in the standard stereographic triangle $0 \leqslant \omega_1 \leqslant \omega_2 \leqslant \omega_3$, yields the following expressions:

(i) for the cube

$$\text{if } \omega_1 = 0 \text{ then } \overline{\psi_V}(\omega) = \frac{1}{4\pi^2}\frac{1}{\omega_2\omega_3^2},$$

$$\text{if } \omega_1 + \omega_2 - \omega_3 > 0 \text{ then } \overline{\psi_V}(\omega) = \frac{1}{12\pi^2}\frac{3\omega_2 - \omega_1}{\omega_2^2\omega_3^2},$$

$$\text{else } \overline{\psi_V}(\omega) = \frac{1}{24\pi^2}\left(\frac{(\omega_3 - \omega_1)^3 + (\omega_3 - \omega_2)^3 + (\omega_2 - \omega_1)^3 + 6\omega_1\omega_2\omega_3 - (\omega_1^3 + \omega_2^3 + \omega_3^3)}{\omega_1^2\omega_2^2\omega_3^2}\right).$$

(ii) for the octahedron

$$\text{if } \omega_1 = \omega_2 = 0 \text{ then } \overline{\psi_V(\omega)} = \frac{1}{\pi^2},$$

$$\text{else } \overline{\psi_V}(\omega) = \frac{1}{\pi^2(\omega_1 + \omega_2)}\left(\frac{(\omega_1 + \omega_2)\omega_3^2 + \omega_1\omega_2\omega_3}{(\omega_1 + \omega_3)^2(\omega_2 + \omega_3)^2(\omega_3 - \omega_1)}\right).$$

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